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ON THE APPLICATION OF THE LAPLACE TRANSFORM TO CERTAIN ECONOMIC PROBLEMS*

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During the last thirty years, the method of the Laplace transform has found an increasing number of applications in the fields of physics and technology. In this article the author points out the possibility of solving problems in the area of discounting with the aid of this method. Without any loss of general validity, it is shown that a discount factor can always be written in an exponential manner which implies that the present value of a cash-flow will obtain a very simple form in the Laplace terminology. This simplicity holds good for stochastic as well as for deterministic economic processes, and the results mentioned below should, therefore, be of immediate use when applied, e.g., to investment problems.

1. Introduction

Very often the economist is confronted with problems of calculating the discounted or cumulated value of a cash flow, the internal rate of interest of an investment project, or similar quantities [2, p. 367]. The methods applied for specific solutions have almost always been of a rather "classical" design, using the calculus and ordinary algebra. It is my intention with this paper to introduce a perhaps more advanced approach with the aid of the Laplace transform, the theory of which is highly developed and which has found a wide variety of applications—solutions of differential equations, electrical network constructions, servo-technic mechanisms, to mention only a few.

Due to the fact that the Laplace theory already exists, there should be immense possibilities for an almost immediate use in stochastic as well as deterministic economic problems—a fact which seems to have been little, or indeed, hardly at all investigated in the literature.

2. Economic Problems

Consider the case of an investment project. For various alternatives one wishes to calculate the present value of a series of cash receipts and disbursements (transactions) $c_1, c_2, c_3 \cdots$ at the future points of time $t_1, t_2, t_3 \cdots$. If we let c_{ij} be the amount of money to be received ($c_{ij} > 0$) or to be payed out ($c_{ij} < 0$) for the ith alternative (out of a set of M alternatives) at the jth point of time, the present value of this alternative takes the form:

$$(1) \qquad V_i = \sum_{j=1}^{N_i} c_{ij}(1 + r_{1i}T)^{-n_{ij}} = \sum_{j=1}^{N_i} c_{ij}(1 + r_{1i}T)^{-t_{ij}/T}$$

where:

r_{1i} = discounted rate of interest.
T = unit time period to which r_{1i} is related.

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$n_{ij} = t_{ij}/T$ = number of unit time periods until the jth transaction for the ith alternative takes place.

N_i = total number of transactions for the ith alternative.

If we choose the present value as a measuring rod for the selection of the best alternative given a certain discount rate, the optimal decision is reached by taking i, such that:

(2) $$V_{\text{opt}} = \text{Max}_i \{V_i\}.$$

If, on the other hand, the maximum internal rate of interest is our aim, we calculate the different r_{1i} by putting $V_i = 0$ in equation (1). The optimum alternative is thus determined by choosing i maximizing the sequence $\{r_{1i}\}$:

(3) $$r_{1\text{ opt}} = \text{Max}_i \{r_{1i}\}.$$

In equation (1) we have chosen a rate of interest related to a unit period of time T, e.g. a year or six months, such that a corresponding capital increment after one unit period of time amounts to the capital multiplied by the rate of interest.

Let us now define a new quantity r_{2i}, such that:

$$e^{r_{2i}T} = (1 + r_{1i}T)$$

or

(4) $$r_{2i} = (1/T) \log (1 + r_{1i}T).$$

Equation (1) becomes:

(5) $$V_i = \sum_{j=1}^{N_i} c_{ij} \exp (-r_{2i}t_{ij})$$

which means that by defining a new rate of interest according to (4), we receive the present value of each alternative as if the transactions were discounted continuously at this second interest rate. This result shows that there exists no loss of generality by employing formula (5) in all problems of this kind as long as the rate of interest definition (4) is used.

Consider now the more general case where receipts and disbursements can exist continuously in time together with transactions at discrete points of time. The cash flow can now for a certain alternative be suitably described by the time function:

(6) $$v(t) = v'(t) + \sum_{k=1}^{K} V_k\delta(t - t_k)$$

where

$v'(t)$ = the difference between the continuous in- and out-payments with the dimension of (money units/time unit).

V_k = cash receipt (>0) or disbursement (<0) at the discrete time point t_k with the dimension of (money units).

$\delta(t - t_k)$ = the Dirac Pulse existing at the time point t_k. (The Dirac Pulse $\delta(t - t_k)$, existing at the point of time t_k, can be defined as a function satisfying the relation:

$$\int_N f(t)\delta(t - t_k)\, dt = f(t_k), \quad \text{if} \quad t_k \varepsilon N$$
$$= 0, \quad \text{if} \quad t_k \not\varepsilon N,$$

where $f(t)$ is an arbitrary time function.)

$K=$ total number of "discrete" transactions.

The contribution from the cash flow in the time interval $(t, t + dt)$ to the present value V is:

$$dV = e^{-rt}v(t)\, dt = (v'(t)\, dt + \textstyle\sum_k' V_k)e^{-rt} \tag{7}$$

where $\sum'$ indicates that the summation is performed over those of the set of transactions $\{V_k\}$ existing in the interval in question and r is the discounted rate of interest according to (4).

The present value becomes:

$$V = \int_N v(t)e^{-rt}\, dt = \int_N [v'(t) + \textstyle\sum_{k=1}^K V_k \delta(t - t_k)]e^{-rt}\, dt \tag{8}$$
$$= \int_N v'(t)e^{-rt}\, dt + \textstyle\sum_{k=1}^K V_k \exp(-rt_k)$$

where N is the domain of the time axis where the transactions take place.

If, for any reason, the same discount rate cannot be used for all transactions, the function $v(t)$ will have to be divided into parts, of which each has the same rate in common. The present value will be the sum of the present values of each part calculated according to (8).

Due to the monotonic character of the logarithm applying the internal rate of interest as the evaluator of the set of alternatives, it makes no difference whether the rate r_{1i} or r_{2i} is used as the ordering will be the same in both cases.

3. The Laplace Transform

Applying the method of the Laplace transform, a time function $f(t)$ is transformed to a complex functon of a complex variable $s = \sigma + i\omega$, where i is the imaginary unit $(-1)^{1/2}$, according to the equation:

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st}f(t)\, dt. \tag{9}$$

$F(s)$ is called the Laplace transform of $f(t)$ and $f(t)$ the inverse transform of $F(s)$. In order that the integral should converge, it is necessary that the function $f(t)$ increases less than or equal to an exponential rate. In the latter case the real part of s has to be chosen greater than the exponential factor of the time function.

There exists a unique one-to-one relation between each time function and its corresponding transform, which implies that given the time function, the transform is uniquely determined and vice versa.

The equation which translates the Laplace transform to its time function follows the expression:

(10) $$f(t) = (1/2\pi i) \int_{c-i\infty}^{c+i\infty} e^{st} F(s)\, ds$$

where the integration takes place over a straight vertical line in the complex s-plane. Due to the rather involved character of equation (10), one usually plots (in accordance with (9)) a series of frequently recurring time functions to the s-domain in a Laplace dictionary, which becomes of use when translating back to the time domain an s-expression received in solving a specific problem.

The usefulness of this method is due to the fact that a great variety of operations on the time function under consideration correspond to rather simple algebraic operations on the transform. Hence, a number of theorems have been developed several of which will be mentioned below.

I. Time Derivation

The transform of the time derivative of a function $f(t)$ follows the equation:

(11) $$\pounds\{df(t)/dt\} = sF(s) - f(0)$$

where of course $F(s)$ is the transform of $f(t)$ and $f(0)$ is the value of $f(t)$ at $t = 0$.

II. Time Integration

The transform of the time integral of $f(t)$ is received through the relation:

(12) $$\pounds\left\{\int_0^t f(\tau)\, d\tau\right\} = (1/s)F(s) + (1/s)\left[\int_0^t f(\tau)\, d\tau\right]_{t\to+0}.$$

III. Translation

The transform of a function starting at the time a when given the transform of the same function starting at the time $t = 0$ is given simply by multiplying the latter transform with the factor e^{-as}, or in mathematical terms:

(13) $$\pounds\begin{Bmatrix} f(t-a), & \text{for} \quad t - a > 0 \\ 0, & \text{for} \quad t - a \leqq 0 \end{Bmatrix} = e^{-as}F(s).$$

IV. The Limit Value Theorems

The following two relations hold good for the limit values of respective functions:

(14) $$\lim_{s\to0} sF(s) = \lim_{t\to\infty} f(t)$$

(15) $$\lim_{s\to\infty} sF(s) = \lim_{t\to0} f(t).$$

V. The Damping Theorem

A time function $f(t)$ multiplied by a damping factor e^{-at} has the transform:

(16) $$\pounds\{e^{-at}f(t)\} = F(s + a).$$

This theorem is of special interest in economic problems since the constant a can be interpreted as the discount rate.

VI. Derivation of a Second Variable

The derivative of a time function $f(t, a)$ with respect to a second parameter a has the transform:

(17) $$\pounds\{(\partial/\partial a)f(t, a)\} = (\partial/\partial a)F(s, a).$$

VII. Integration over a Second Variable

The integral of a time function $f(t, a)$ over an interval of a second parameter a has the transform:

(18) $$\pounds\left\{\int_{a_1}^{a_2} f(t, a)\,da\right\} = \int_{a_1}^{a_2} F(s, a)\,da.$$

This relation may be of interest when applied to a stochastic process where a is a stochastic variable.

With regard to the above-mentioned Laplace dictionary, a small fraction of "ordinary" time functions and their corresponding transforms are included in the following table:

$\pounds\{\delta(t)\}$	$= 1$	Dirac Pulse
$\pounds\{1\}$	$= 1/s$	Step Function
$\pounds\{t\}$	$= 1/s^2$	Ramp Function
$\pounds\{e^{-at}\}$	$= 1/(s + a)$	
$\pounds\{\cos at\}$	$= s/(s^2 + a^2)$	
$\pounds\{e^{-bt}\cos at\}$	$= (s + b)/((s + b)^2 + a^2).$	

4. Application of the Laplace Transform to Deterministic Economic Processes

In a deterministic economic process the value of each function under consideration is known in advance at every future point of time. In specific problems such functions can be the number of demanded units per unit time period, cost per unit time period, etc.

If we use the idea of maximizing the discounted value of a future cash flow existing during a given time interval (finite or infinite) as the main goal of a company, this value will take a very simple form in Laplace terms. We denote this cash flow function by $v(t)$ (as in (8)), letting the integration domain become the whole of the positive time axis, as all contributions to the integral vanish where $v(t)$ does not exist. Our discounted value is:

(19) $$V = \int_0^\infty e^{-rt}v(t)\,dt = \lim_{t\to\infty}\int_0^t e^{-r\tau}v(\tau)\,d\tau.$$

Using equations (12) and (14), at the same time assuming the integral to be continuous at $t = 0$, we obtain:

(20) $$V = \lim_{s\to 0} s(1/s)\pounds\{e^{-rt}v(t)\}.$$

This expression can, however, be greatly simplified with the aid of the damping theorem. Denoting the transform function of $v(t)$ by $v^*(s)$, this theorem yields:

$$V = \lim_{s \to 0} v^*(s + r) = v^*(r) = [\pounds\{v(t)\}]_{s=r} . \tag{21}$$

This result is of great interest as it gives *a palpable economic interpretation of the Laplace transform*, namely as the present value of a future cash flow.

If we divide $v(t)$ into the two functions $R(t)$ (= revenues) and $P(t)$ (= out-payments), equation (21) becomes:

$$V = [\pounds\{R(t)\}]_{s=r} - [\pounds\{P(t)\}]_{s=r} . \tag{22}$$

The internal rate of interest is obtained by solving r in the expression:

$$[\pounds\{R(t)\}]_{s=r} - [\pounds\{P(t)\}]_{s=r} = 0. \tag{23}$$

In the case where the time function of the cash flow for all future points of time is known as a mathematical expression $g(t)$, but we only wish to study the discounted value from a finite time interval $[0, T]$, the Laplace expression becomes somewhat more involved. The cash flow function we wish to discount can then be written as:

$$\begin{aligned} f(t) &= g(t), \quad \text{for} \quad t \leqq T \\ &= 0, \quad \text{for} \quad t > T. \end{aligned} \tag{24}$$

Using equation (13), we receive:

$$V = [\pounds\{f(t)\}]_{s=r} = [\pounds\{g(t)\} - e^{-sT}\pounds\{g(t + T)\}]_{s=r} . \tag{25}$$

The method of obtaining V in this case will consist of the following steps:

1. Translate $g(t)$ into its s-function.
2. Compute $g(t + T)$ by substituting t for $t + T$.
3. Translate $g(t + T)$ into its s-function.
4. Use equation (25) with $s = r$.

An alternative way of finding the value V, would be by using the damping theorem:

$$V = [\pounds^{-1}\{(1/s)G(s + r)\}]_{t=T} \tag{26}$$

where $\pounds^{-1}$ denotes the inverse transformation.

Consider the following case, the functions existing during an infinite or finite future time interval.

$p(t)$ = amount of produced units per unit time period.

$d(t)$ = amount of demanded units per unit time period.

$i(t)$ = amount of inventory units.

c = fixed costs per unit time period.

a_p = (variable) production and sales cost per unit.

a_d = market price of each unit.

a_i = inventory cost per unit and unit time period.

We assume c, a_p, a_d, a_i are independent of time and of the amounts $p(t)$, $d(t)$, $i(t)$.

The present value of the discounted profit, assuming all other than the mentioned revenues and costs to give negligible contributions, will be:

$$V = \int_0^\infty e^{-rt}[a_d d(t) - a_p p(t) - a_i i(t) - c]\, dt \tag{27}$$

or in the Laplace language:

$$V = a_d[\pounds\{d(t)\}]_{s=r} - a_p[\pounds\{p(t)\}]_{s=r} - a_i[\pounds\{i(t)\}]_{s=r} - [\pounds\{c\}]_{s=r}\,. \tag{28}$$

If the inventories are i_0 units at $t = 0$, $i(t)$ is determined by:

$$i(t) = i_0 + \int_0^t [p(\tau) - d(\tau)]\, d\tau \tag{29}$$

if the demand can always be satisfied. Hence $\pounds\{i(t)\}$ will become:

$$\pounds\{i(t)\} = \pounds\{i_0\} + (1/s)[\pounds\{p(t)\} - \pounds\{d(t)\}] \tag{30}$$

which, if introduced into (28), will simplify this expression:

$$V = (a_d + a_i/r)[\pounds\{d(t)\}]_{s=r} - (a_p + a_i/r)[\pounds\{p(t)\}]_{s=r} - a_i[\pounds\{i_0\}]_{s=r} - [\pounds\{c\}]_{s=r}\,. \tag{31}$$

This equation is of great use when applied to, e.g., a feedback controlled production or to inventory problems.

5. Application of the Laplace Transform to Stochastic Economic Processes

In a stochastic economic process, the functions studied possess some property of giving uncertain outcomes. This uncertainty is due to the influence of one or a set of stochastic variables assuming values which are not known in advance. In many cases one presupposes the existence of a probability density function (p.d.f.) describing the probability of these variables taking certain exact values or values within a certain interval. If the p.d.f. for each stochastic variable is time invariant, the process considered is said to be stationary. In this paper it will be assumed, only for the sake of simplicity, that the process studied is influenced by a single stochastic variable.

When studying stochastic economic processes, there exists no unique measure of optimality [1, p. 38] for deciding the best of a set of alternatives, i.e., if the main goal of a company is given as a function of economic variables (e.g. the present value of a cash flow), one still has to choose some norm to use when comparing the different possible outcomes of the function considered.

The measure for this use, most frequently mentioned in literature, is the *expected value* of the function concerned. Given the p.d.f. of the stochastic variable X as $f(x)$, the expected value of a function $\phi(X)$ depending on X is determined by:

$$E\{\phi(X)\} = \int_{-\infty}^{\infty} f(x)\phi(x)\,dx \tag{32}$$

where the integration is performed over all possible outcomes of $\phi(X)$.

If ϕ describes a stochastic process, ϕ will assume different values at different points of time. It is therefore suitable to include time as a second variable in the functional notation, i.e. $\phi(X, t)$. For a non-stationary process the p.d.f. will also depend on the time, hence $f(x, t)$.

If we now let $\phi(X, t)$ mean a cash flow function depending on the stochastic variable X, which, e.g., can be the number of units sold per unit time period, we denote:

$$v(X, t) = \phi(X, t). \tag{33}$$

The expected present value of $v(X, t)$ is obtained by:

$$E\{V\} = \int_{-\infty}^{\infty} dx \int_0^{\infty} f(x, t)v(x, t)e^{-rt}\,dt. \tag{34}$$

Using the Laplace notation this becomes:

$$E\{V\} = \int_{-\infty}^{\infty} [\pounds\{f(x, t)v(x, t)\}]_{s=r}\,dx \tag{35}$$

or

$$E\{V\} = \left[\pounds\left\{\int_{-\infty}^{\infty} f(x, t)v(x, t)\,dx\right\}\right]_{s=r} dx \tag{36}$$

where equation (18) (theorem VII) has been applied. The integrand in (36) is, of course, the mean value of $v(X, t)$, which for a non-stationary process will be time dependent:

$$E\{v(X, t)\} = m(t) = \int_{-\infty}^{\infty} f(x, t)v(x, t)\,dx \tag{37}$$

and

$$E\{V\} = [\pounds\{m(t)\}]_{s=r}. \tag{38}$$

For the case that $v(X, t)$ is stationary, the mean value (37) will be constant and:

$$E\{V\} = [\pounds\{m\}]_{s=r}. \tag{39}$$

If this stationary process terminates at $t = T$, (39) becomes simply:

$$E\{V\} = (m/r)(1 - e^{-rT}). \tag{40}$$

It should perhaps be emphasized that if $X(t)$ is stationary, $v(X, t)$ is not necessarily stationary and vice versa.

In short, it can be said that the main result of this paragraph is that stochastic economic processes can be treated in exactly the same manner as deterministic processes if we use the (time depending) mean value of the function studied. This is of course valid under the assumption of using the expected value as a suitable measure of the process result.

Example

Compute the expected present value of a season's variational revenue flow. The demand is described by:

$$d(X, t) = X \cos \omega t + A \qquad \text{(units/unit time period)}$$

where:

$$2\pi/\omega = \text{period of a complete season's fluctuation (time units)}$$

$$X = \text{stochastic amplitude of fluctuation}$$

$$A = \text{constant}$$

and given:

$$f(x) = \text{p.d.f. of the stochastic amplitude}$$

$$a_d = \text{market price of each unit (constant)}.$$

We assume that the demand is non-negative, i.e. $|X| \leq A$ for every possible X, and also that X is stationary.

Denoting the revenue flow by $R(X, t)$, we obtain:

$$R(X, t) = a_d(X \cos \omega t + A).$$

The mean value of this function is:

$$m_R(t) = \int_{-\infty}^{\infty} R(x, t)f(x)\, dx = a_d(m_X \cos \omega t + A)$$

where m_X is the mean value of X. Note that R is non-stationary as its mean value is time dependent while X is stationary.

From the Laplace dictionary included in this paper we obtain:

$$[\pounds\{\cos \omega t\}]_{s=r} = r/(r^2 + \omega^2).$$

Using equation (38) and assuming that the time period considered is infinite, the expected present value of R becomes:

$$[\pounds\{a_d(m_X \cos \omega t + A)\}]_{s=r} = a_d(rm_X/(r^2 + \omega^2) + A/r)$$

which is the result we seek.

6. Final Remarks

This paper makes no claim to be complete, but its aim is to attempt to show some simple results, which, I believe, should be of great use when applied to the class of economic problems described above. It seems that extensions could easily be made to nearby fields of application.

The use of the Laplace transform in these problems has its limits, e.g., the integrals involved are assumed to converge; economic simplifications are introduced, etc. Such, however, applies to all mathematical simplifications of reality.

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