Stochastic and Deterministic Population Dynamics

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Outline

This talk will shortly consider:

• **Problem presentation**
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- Comparison between different approaches
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END
Problem presentation I

Population dynamics describes the evolution in time of populations.

\[ X_\alpha(t) \in \mathbb{Z}^+, \alpha = 1 \ldots N \]

non-negative integer population sizes at time \( t \in \mathbb{R}^+ \). The time-dependent problem is described by a jump-process in which the events \( j = 1 \ldots E \) at random times alter the population numbers in a prescribed form:

\[ X_\alpha \rightarrow X_\alpha + \delta_j^\alpha. \]
Problem presentation II

The random events are exponentially distributed with rates $W_j(X(t))$, the process is described by the integral equation

$$X_\alpha(t) = X_\alpha(0) + \sum_{j=1}^{E} Y\left(\int_{s=0}^{t} W_j(X(s))ds\right)$$

where $Y$ are independent Poisson processes (Kurtz 1981).

This process corresponds precisely to a Feller process (equivalently, think of Monte Carlo simulations).
Markov Jump Processes

Jump processes have an embedded discrete process that maps into the continuous time process by the exponentially distributed event times. We can work on the discrete side most of the time.

Given the jump process $X(t)$ with events $j = 1, \ldots, E$ and rates $W_j(X(t))$ the time for the next event is exponentially distributed with mean $1/(\sum_{j=1}^{E} W_j(X(t)))$.

The selected event is $j$ with probability

$$P(j) = \frac{W_j(X(t))}{\sum_{k=1}^{E} W_k(X(t))}.$$
Deterministic (rate) equation

The following heuristic procedure gives (mean field equation):

\[
dX_\alpha(t)/dt = \lim_{\varepsilon \to 0} E(\Delta^\alpha|X(t - \varepsilon)) = \sum_{j=1}^{E} \delta_j^\alpha W_j(X(t))
\]

where the expectation values are taken with the conditional probability of the Markov process.

Notice that we have transformed a vector in \((\mathbb{Z}^+)^N\) into a vector in \((\mathbb{R}^+)^N\).
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• In which form?
• does it represent the evolution of averages?
• does it contain relevant information of the Feller process?
• how can we use this information?
Comparison I

It is not the evolution equation of averages.
The relation among various approximations and related processes is:

**Comparison II**

Notice that SDE is a replacement of the original problem, not an approximation.
Theorem K1 (Deterministic limit): If
\[ W_j(X(t)) = \Omega w_j(X(t)/\Omega) \] (mass-action law), then
\[ \lim_{\Omega \to \infty} X(t)/\Omega = x(t) \]
satisfies the (limit of) the deterministic equation, i.e.,
\[ dx/dt = \sum_{j=1}^{E} \delta_j w_j(x). \]

Theorem K2 (Central limit): For any fixed \( t \),
\[ \lim_{\Omega \to \infty} (X(t) - \Omega x(t))/\sqrt{\Omega} \]
is a Wiener (Langevin) process.
Details: Convergence is not uniform unless $\tau \leq t \leq T$.

Both results must be used away from extinction values of populations.

(There should be an improved version of Theorem 2 restoring the integer nature of populations).
Sustained Oscillations I

Stochastic stability theory: Kushner (1967, 1972). $M_n$ is (super) sub-martingale if the conditional expectation with respect to the process $X_n$ satisfies:

$$E(M_{n+1}|X_n = x_n, \ldots X_0 = x_0)(\leq) \geq M_n.$$ 

**Theorem 1**: If the deterministic system has a (non extinction) linearly stable fixed point, and the mass action law applies, the Lyapunov functions, $H$, of the deterministic system determine two concentric regions in the stochastic system, around the equilibrium.
Sustained Oscillations II

- Inner region: $H$ increases with time. Sub-martingale, transient.
- $N = 2$ and complex eigenvalues: the phase of the system increases (sustained oscillations).

**Theorem 2**: If the deterministic system has an extinction stable fixed point ($\sum_j W_j(0) = 0$), the mass-action law applies, and $w_j(x) = O(x)$ at $x = 0$.

The total population is a Lyapunov function and the stochastic system is stable whenever the deterministic system is linearly stable.
Simplest epidemic model:

<table>
<thead>
<tr>
<th>Event</th>
<th>Effect</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>Birth</td>
<td>((N, n) \rightarrow (N + 1, n))</td>
<td>(W^b \equiv a/\Omega)</td>
</tr>
<tr>
<td>Contagion</td>
<td>((N, n) \rightarrow (N - 1, n + 1))</td>
<td>(W^c \equiv \beta Nn/\Omega)</td>
</tr>
<tr>
<td>Death</td>
<td>((N, n) \rightarrow (N, n - 1))</td>
<td>(W^d \equiv bn/\Omega)</td>
</tr>
</tbody>
</table>

Deterministic equilibrium at:

\[ N_{eq} = \Omega b/\beta, \ n_{eq} = \Omega a/b \]

Lyapunov functions \(0 \leq \Gamma \leq 1\):

\[ E_\Gamma = ((1 - \Gamma)x^2 + \alpha^2 y^2) + \Gamma (x + y)^2 \]

\[ \alpha^2 \equiv \frac{b^2}{\beta a} = \frac{N_{eq}}{n_{eq}} > \frac{1}{4}. \]
Example II

Example III

Example IV

Consider the jump process $X_n$ with $X_0 = X_{eq}$, in the vicinity of the point $X_{eq} = (N_{eq}, n_{eq})$. Let $Y = X - X_{eq}$, $H(Y) = \sum_{i,j=1}^{N} L_{ij}((X_i - X_{eq}^i)(X_j - X_{eq}^j)$ and $M_n = H(Y_n)$.

Let $E_0 = E(H(Y_1|Y_0 = 0)$. Then by direct computation:

$$E(M_{n+1} - M_n|X_n \ldots X_0) =$$

$$\sum_{i,j=1}^{N}((X_i D_j(X) + D_i(X)X_j)L_{ij}) + E_0 + O((X/\Omega)^2)$$

where $D_j$ is the linearized vector field of the deterministic equation. Moreover, $E_0 \geq 0$ and

$$DH = \sum_{i,j=1}^{N}((Y_i D_j(Y) + D_i(Y)Y_j)L_{ij}) \leq 0.$$
Example V

Hence,

$$E(M_{n+1} - M_n | X_n \ldots X_0) = 0$$

separates the phase space in a super and a sub-martingale region. The sub-martingale is a topological disk centered at $X_{eq}$, the super-martingale is in an annulus.

Strong law of large numbers: The system oscillates back and forth from one region to the other. In the super-martingale region we further have that $P(M_n \geq M_0 b) \leq 1/b$ for any trajectory not leaving the super-martingale region (Markov inequality applied to super-martingales). A better bound can be achieved if $DH$ is bounded.
Conclusions I

• We have successfully used the approximations of linear stability theory in smooth dynamical systems in the framework of stochastic jump processes.
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• The framework presented consists in approximating stochastic processes without leaving the phase space of the original process, relying on limit cases or substituting the original process.

• The stochastic blow-up of the stable fixed point shows that it splits in an inner unstable region immersed in a larger stable region.
Conclusions II

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