Exercise 1

The purpose of these exercises is to become familiar with the concept of vector and matrix norms and singular value decomposition. Most of the exercises are theoretical but some of them require that you write a program in MATLAB or Python.

Hand-in your results electronically latest Sept. 12, 2018, 24:00h.

This lab has 6 tasks.

Task 1

(cf. Exercise 3.2 in the course book). Let $A$ be a square matrix, $\rho(A)$ be the spectral radius of $A$ and let $\|\cdot\|$ any operator norm. Show

$$\rho(A) \leq \|A\|$$

Task 2

Use the statement in Task 1 to show

$$\|A\| < 1 \Rightarrow \lim_{n \to \infty} A^n = 0$$

where $A$ is a square matrix, 0 the zero matrix of corresponding dimensions and $\|\cdot\|$ an operator norm.

Task 3

This task is related to the equivalence of norms theorem. Show

1. $\|x\|_\infty \leq \|x\|_2$
2. $\|x\|_2 \leq \sqrt{n}\|x\|_\infty$
3. $\|A\|_\infty \leq \sqrt{n}\|A\|_2$
4. $\|A\|_2 \leq \sqrt{n}\|A\|_\infty$
For $A$ being an $n \times n$ matrix and $x$ being an $n$ vector.

**Task 4**

Consider a row matrix $A$ and a vector $a$ with the same entries as $A$. In the lecture notes we showed that $\|A\|_2 = \|a\|_2$, where we use on the left side of the equation the matrix norm and on the right hand side the vector norm. What about the 1- and $\infty$-norm? Can one prove a similar theorem or can you find a counter example?

**Task 5**

Let $u, v$ be two $n$ vectors. Show that their outer product is an $n \times n$ matrix of rank 1.

**Task 6**

Let $uv^T$ be the outer product of two vectors. Give its singular value decomposition in terms of these two vectors. Make also a numeric example and verify your statement in MATLAB/Python. (for Python you have to import scipy.linalg)