Exercise 3

Hand-in your results electronically latest Oct. 06, 2018, 24:00h.
This lab has 4 tasks.

Task 1

The condition number of a matrix gives a sharp estimate of the sensitivity of $x$ with respect to perturbations of $b$ when solving $Ax = b$, this means there exists a right hand side $b$ and a perturbation $\delta b$ such that

$$\frac{\|\delta x\|_2}{\|x\|_2} = \kappa_2(A) \frac{\|\delta b\|_2}{\|b\|_2}$$

(Note the equal sign!). Give a vector pair $(b, \delta b)$ for which this equality holds. Hint, express these vectors in terms of left singular vectors.

Task 2

Hilbert matrices are notoriously ill conditioned. Verify your result from Task 3 by solving a linear system with a $50 \times 50$ Hilbert matrix and a worst case $b$ and $\delta b$. Hilbert matrices and their exact inverses can be constructed in MATLAB by `hilb` and `invhilb` and in Python by the commands `scipy.linalg.hilbert` and `scipy.linalg.invhilbert`.

Task 3

Let

$$A = \begin{pmatrix} a_{11} & w^T \\ w & A_1 \end{pmatrix}$$

be a $n \times n$ positive definite matrix. Show that $a_{11}$ is strictly positive and that the $(n - 1) \times (n - 1)$ submatrix is positive definite.
**Task 4**

Show that a strictly diagonally dominant matrix $A$ is invertible. Hint: Show that there exists no vector $u \neq 0$ which solves $Au = 0$. For this assume $u \neq 0$ and show that $u$ cannot have all elements of the same size (in absolute value) and that furthermore there is no element $u_i$ with $|u_i| \geq |u_j|$ for all $j$.

**Task 5**

There are lots of 'islands' in the world-wide-web, meaning clusters of websites that are not connected to other parts of the world wide web via hyperlinks. Assume that there are $r$ different clusters.

Prove: The dimension of the eigenspace to the eigenvalue 1 of the hyperlink matrix $H$ is then at least $r$.

You might consider as an example the disconnected hyperlink graph

![Disconnected Hyperlink Graph](image)

It has the hyperlink matrix

$$H = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 1 & 0 \\
\end{pmatrix}$$

This matrix has then (according what you have to show) $r = 2$ eigenvalues $\lambda = 1$ with two linearly independent eigenvectors.