Exercise 4

The purpose of these exercises is to train more on various orthogonalization techniques.

Hand-in your results electronically latest Oct. 08, 2014, 24:00h.

This lab has 5 tasks.

Task 1

Let

\[
A = \begin{pmatrix}
a_{11} & w^T \\
w & A_1
\end{pmatrix}
\]

be a \( n \times n \) positive definite matrix. Show that \( a_{11} \) is strictly positive and that the \( (n-1) \times (n-1) \) submatrix is positive definite.

Task 2

Show that a strictly diagonally dominant matrix \( A \) is invertible. Hint: Show that there exists no vector \( u \neq 0 \) which solves \( Au = 0 \). For this assume \( u \neq 0 \) and show that \( u \) cannot have all elements of the same size (in absolute value) and that furthermore there is no element \( u_i \) with \( |u_i| \geq |u_j| \) for all \( j \).

Task 4

Consider the matrix

\[
A = \begin{pmatrix}
5 & 0 & 0 & -1 \\
1 & 0 & -3 & 1 \\
-1.5 & 1 & -2 & 1 \\
-1 & 5 & 3 & -3
\end{pmatrix}
\]

and the matrix \( A(p) \) with the same diagonal elements as \( A \) and all other elements being scaled by \( p \in [0,1] \), i.e. \( pA_{ij} \). Note, \( A(0) \) is a diagonal matrix. Compute the eigenvalues of \( A(p) \) with \texttt{eig} and plot them in the complex plane. Vary the parameter \( p \) so that you trace in the plot their dependency on \( p \). Mark in the plot the diagonal elements of \( A \) and also the eigenvalues of \( A \) with fat circles. Mark in the same plot the Gerschgorin
discs and explain by your figure Gerschgorin’s theorem.

Task 5

This task refers to the lecture on Wednesday, Sep. 20, 2017. There are lots of ‘islands’ in the world-wide-web, meaning clusters of websites that are not connected to other parts of the world wide web via hyperlinks. Assume that there are \( r \) different clusters.

Prove: The dimension of the eigenspace to the eigenvalue 1 of the hyperlink matrix \( H \) is then at least \( r \).

You might consider as an example the disconnected hyperlink graph

\[
\begin{array}{cccc}
A & B & C & D \\
\end{array}
\]

It has the hyperlink matrix

\[
H = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\
0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\
0 & 0 & \frac{1}{2} & \frac{1}{2} & 0
\end{pmatrix}
\]

This matrix has then (according what you have to show) \( r = 2 \) eigenvalues \( \lambda = 1 \) with two linearly independent eigenvectors.