Exercise 4

The purpose of these exercises is to train eigenvalue methods.
Hand-in your results electronically latest Oct. 11th, 2018, 24:00h.

This lab has 4 tasks.

Task 1

Implement the QR method with Rayleigh shifts and deflations (Alg. 28.2). Test it on several symmetric matrices and compare the result with \texttt{eig}. Test it even on a symmetric orthogonal matrix. Apply it on many random symmetric matrices and make a statement about the average number of iterations you need to get the eigenvalues with a relative error \( \leq 1.e^{-8} \).

Note a matrix given by

\[
A := \text{rand}(n,n); \quad A := \frac{1}{2}(A + A^T)
\]

is a random symmetric matrix.

Task 2

Solve Exercise 25.1 in the book. Please note the hint given there. Explain how the statement \( \text{rank}(A - \lambda I) \geq m - 1 \ \forall \ \lambda \in \mathbb{C} \) is related to the statement you should prove.

Task 3 - a challenge!

Study the description of the bisection algorithm on pp. 227-229 of the course book. Implement the method and test it on matrices which are in tridiagonal form. You might take a general symmetric matrix and transform it to Hessenberg form with the Python command \texttt{scipy.linalg.hessenberg} or the matlab command \texttt{hessn}.
Task 4

Consider the matrix

\[ A = \begin{pmatrix} 5 & 0 & 0 & -1 \\ 1 & 0 & -3 & 1 \\ -1.5 & 1 & -2 & 1 \\ -1 & 5 & 3 & -3 \end{pmatrix} \]

and the matrix \( A(p) \) with the same diagonal elements as \( A \) and all other elements being scaled by \( p \in [0, 1] \), i.e. \( pA_{ij} \). Note, \( A(0) \) is a diagonal matrix. Compute the eigenvalues of \( A(p) \) with \texttt{eig} and plot them in the complex plane. Vary the parameter \( p \) so that you trace in the plot their dependency on \( p \). Mark in the plot the diagonal elements of \( A \) and also the eigenvalues of \( A \) with fat circles. Mark in the same plot the Gerschgorin discs and explain by your figure Gerschgorin’s theorem.