Numerical Linear Algebra
Unit 1: Matrix and Vector Norms

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Vector Norms

Matrix Norms
Norm Axioms

Definition

A vector norm \( \| \cdot \| \) is defined by \( \| \cdot \| : \mathbb{R}^n \to \mathbb{R} \) with

- \( \| x \| \geq 0 \)
- \( \| x \| = 0 \iff x = 0 \)
- \( \| x + y \| \leq \| x \| + \| y \| \) (triangular inequality)
- \( \| \alpha x \| = |\alpha| \| x \| \quad \alpha \in \mathbb{R} \)
Examples

Norms we use in this course:

\[ \|x\|_p := (|x_1|^p + \ldots + |x_n|^p)^{1/p} \]

so-called \textit{p-norm} or \textit{Hölder-norm}.

Example

\[ \|x\|_1 = |x_1| + \ldots + |x_n| \]
\[ \|x\|_2 = \left( |x_1|^2 + \ldots + |x_n|^2 \right)^{1/2} \text{ (Euklid)} \]
\[ \|x\|_\infty = \max |x_i| \]

Weighted norms:

\[ \|x\|_W := \|Wx\| \]

\(W\) nonsingular, often diagonal. Used in certain proofs and in statistics to weight the different reliability of measurements.
Some inequalities

\[ \|x\| - \|y\| \leq \|x - y\| . \] This can be deduced from the triangular inequality:

\[ \|x\| = \|(x - y) + y\| \leq \|(x - y)\| + \|y\|. \]

For the 2-norm we have the Cauchy-Schwarz inequality:

\[ |(x, y)| \leq \|x\|\|y\| \]
Equivalence of Norms in finite dimensions

Theorem

All norms on $\mathbb{R}^n$ are equivalent in the sense:
There are constants $c_1, c_2 > 0$ such that for all $x$

$$c_1 \|x\|_\alpha \leq \|x\|_\beta \leq c_2 \|x\|_\alpha$$

holds.

Proof in three steps:
First we show that if two norms are equivalent to $\|\cdot\|_1$ then they are equivalent to each other (transitivity).

Next we note

$$c_1 \|x\|_1 \leq \|x\|_\beta \leq c_2 \|x\|_1 \quad \forall x \neq 0 \iff c_1 \leq \|u\|_\beta \leq c_2 \quad \forall u \text{ with } \|u\|_1 = 1$$

Finally we show that

$$f : x \mapsto \|x\|_\beta$$

is continuous with respect to the $\|\cdot\|_1$ norm. The rest follows from the fact, that a continuous function takes its maximum and minimum on a compact set, here the unit sphere.
Vector Norms

Matrix Norms
Matrix norms as operator norms

We consider a matrix as a linear operator:

\[ A : x \in \mathbb{R}^n \mapsto y \in \mathbb{R}^m \]

We define a matrix norm by relating the size of the image to the size of its preimage:

**Definition**

A matrix norm can be defined by

\[
\| A \|_{(p,q)} := \sup_{x \in \mathbb{R}^n / \{0\}} \frac{\| Ax \|_p}{\| x \|_q} = \max_{x \in S_q} \| Ax \|_p
\]

with \( S_q := \{ x \in \mathbb{R}^n : \| x \|_q = 1 \} \) (unit sphere).

We note \( Q \) orthogonal \( \rightarrow \| Q \|_{(2,2)} = 1 \).
Special matrix norms

Theorem

- $\|A\|_1 = \|A\|_{(1,1)} = \max_j \sum_i \|A_{ij}\|$ \textit{1-norm}
- $\|A\|_\infty = \|A\|_{(\infty,\infty)} = \max_i \sum_j \|A_{ij}\|$ \textit{\infty-norm}
- $\|A\|_2 = \|A\|_{(2,2)} = \sqrt{\rho(A^T A)}$ \textit{2-norm}

with $\rho$ being the \textit{spectral radius}:

$$\rho(B) = \max_i |\lambda_i(B)|$$ \textit{largest eigenvalue}

(Proof: first two statements in the exercises, third exercise in lecture)
Examples

- Let $A$ be a $1 \times n$ matrix. Let $a$ be a vector with the same entries. Compare the 2-norm of $A$ (matrix norm) with the 2-norm of $a$ (vector norm). (Hint: Apply Cauchy-Schwarz)

- Let $u, v$ be two vectors: $A := uv^T$ is called its outer product. Compute its 2-norm.
Frobenius Norm

Definition

The Frobenius norm of a matrix $A$ is

$$\|A\|_F = \left( \sum_i \sum_j |A_{ij}|^2 \right)^{\frac{1}{2}}$$

it is not an operator norm.

Equivalent expressions

- $\|A\|_F = \left( \sum_i |A_i| \frac{2}{2} \right)^{\frac{1}{2}}$
- $\|A\|_F = \left( \sum_j |A_j| \frac{2}{2} \right)^{\frac{1}{2}}$
- $\|A\|_F = \left( \text{tr}(A^T A) \right)^{\frac{1}{2}} = \left( \text{tr}(AA^T) \right)^{\frac{1}{2}}$
Bounding products of matrices

**Theorem**

\[ \|AB\|_F \leq \|A\|_F \|B\|_F \]

**Proof:**
Consider \( C := AB \) and note \( C_{ij} = (A_i, B_j) \).

Apply Cauchy-Schwartz inequality:

\[ |C_{ij}|^2 \leq |A_i|^2 |B_j|^2 \]

and sum up

\[ \|C\|_F^2 = \sum_i \sum_j |C_{ij}|^2 \leq \sum_i |A_i|^2 (\sum_j |B_j|^2) = \|A\|_F^2 \|B\|_F^2 \]
Invariance under Orthogonal Multiplication

A matrix $Q$ is orthogonal if $Q^T Q = QQ^T = I$

**Theorem**

*The 2-norm and the Frobenius norm are invariant under multiplication with orthogonal matrices, i.e.*

$$\|QA\|_2 = \|A\|_2 \text{ and } \|QA\|_F = \|A\|_F$$

The proof uses in the Frobenius case the representation of the Frobenius norm by the trace operator.