Numerical Linear Algebra
Unit 3: More on SVD

Numerical Analysis, Lund University

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Change of Bases

Let \( A \in \mathbb{R}^{m \times n} \) be

\[
A = U \Sigma V^T
\]

Consider

\[
b = Ax
\]

Let \( \bar{b} = U^T b \) and \( \bar{x} = V^T x \) then

\[
\bar{b} = \Sigma \bar{x}
\]

Facit: There exist bases in both spaces, \( \mathbb{R}^m \) and \( \mathbb{R}^n \) so that \( A \) becomes a diagonal matrix.
Solution of an overdetermined linear system

Find $x \in \mathbb{R}^n$ such that

$$Ax = b$$

is equivalent to

Find $\bar{x} \in \mathbb{R}^n$ such that

$$\Sigma \bar{x} = \bar{b}$$

Case $m > n$ and $A$ full-rank

$$\Sigma = \begin{pmatrix} \Sigma_1 \\ 0_{(m-n) \times n} \end{pmatrix}$$

There exists a solution if and only if

$$\bar{b} = \begin{pmatrix} \bar{b}_1 \\ 0 \end{pmatrix}$$
Solution of an underdetermined linear system

Find \( x \in \mathbb{R}^n \) such that

\[ Ax = b \]

is equivalent to

Find \( \bar{x} \in \mathbb{R}^n \) such that

\[ \Sigma \bar{x} = \bar{b} \]

Case \( m < n \) and \( A \) full-rank

\[ \Sigma = \begin{pmatrix} \Sigma_1 & 0_{m \times n-m} \end{pmatrix} \]

The (nonunique) solution is

\[ \bar{x} = \begin{pmatrix} \Sigma_1^{-1} \bar{b} \\ t \end{pmatrix} \]

with an arbitrary vector \( t \in \mathbb{R}^{n-m} \). Thus the space of solutions has dimension \( n-m \).
SVD and eigenvalues

Differences of the two diagonalizations

<table>
<thead>
<tr>
<th>svd</th>
<th>eig</th>
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<tbody>
<tr>
<td>$A = U \Sigma V^T$</td>
<td>$A = V \Lambda V^{-1}$</td>
</tr>
<tr>
<td>$m \times n$</td>
<td>$n \times n$</td>
</tr>
<tr>
<td>two orthogonal bases</td>
<td>one (common) general basis</td>
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<tr>
<td>exists always</td>
<td>exists only for nondefective matrices, meaning columns of $V$ are basis of $\mathbb{R}^n$</td>
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Properties

**Theorem**

\( A \) has \( r \) nonzero singular values iff \( \text{rank} \ A = r \).

Proof: Write \( A \) in its singular value decomposition and note that \( U \) and \( V \) have full rank.

**Theorem**

Let \( \text{rank} \ A = r \), then \( \text{range}(A) = \text{span}(u_1, \ldots, u_r) \) and \( \text{null}(A) = \text{span}(v_{r+1}, \ldots, v_n) \).

Proof: (Lecture)

**Theorem**

\[\|A\|_2 = \sigma_1 = \max \sigma_i \quad \text{and} \quad \|A\|_F = \sqrt{\sum \sigma_i^2}.\]

Proof: The two norm result follows from the existence proof, the Frobenius norm result follows from the representation of the Frobenius norm by the trace and its invariance under orthogonal transformations.
Theorem

The singular values of $A$ are the square roots of the eigenvalues of $A^T A$ if $m > n$ otherwise of $AA^T$.

Proof: Show that $A^T A$ is similar to $\Sigma^T \Sigma$.

Theorem

The singular values of a symmetric matrix are the absolute values of its eigenvalues.

Proof: Note, $A = Q \Lambda Q^T = Q|\Lambda| \text{sign}(\Lambda) Q^T$. Set $U = Q$ and $V^T = \text{sign}(\Lambda) Q^T$.

Theorem

Let $A$ be a square matrix. Then $|\det(A)| = \prod_i \sigma_i$ holds.

Proof: (Lecture)
Low rank matrices

In Exercise 1.4 you proved that $uv^T$ is a matrix of rank one.

**Theorem**

Let $A = U\Sigma V^T$ be an $m \times n$ matrix of rank $r \leq \min(m, n)$. Then

$$A = \sum_{j=1}^{r} \sigma_j u_j v_j^T$$

**Proof:** direct calculation, makes use of writing $\Sigma$ as a sum of diagonal matrices with only one nonzero entry.

**Theorem**

Let $A_\nu = \sum_{j=1}^{\nu} \sigma_j u_j v_j^T$ with $\nu \leq r$. Then,

$$\|A - A_\nu\|_2 = \inf_{\text{rank } B \leq \nu} \|A - B\|_2 = \sigma_{\nu+1} \text{ (or 0)}$$

Similar result for $\|\cdot\|_F$. 