Numerical Linear Algebra
Unit 3a: Compression using the SVD

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Exploiting the best rank $r$ approximation

- The Theorem says that the largest $r$ singular values give you the best rank $r$ approximation to a matrix in $\| \cdot \|_2$ and $\| \cdot \|_F$.
- This has countless applications!
- Here two examples
Image compression

- A digital image is formed of pixels, these can be arranged in a quadratic array.
- Simplest form is that in each pixel, grayscale values are stored (0 for black, 1 for white, in between grey).
- Thus have rectangular matrix $A \in \mathbb{R}^{m \times n}$.
- Requires to store $m \cdot n$ values.
- Formats such as jpeg use mathematical techniques to compress the image.
The following algorithm is not state of the art in image compression, but gives you a very good idea of potential applications of the theorem.

Idea for compression using the SVD and theorem:

1. Make image to obtain $A$
2. Do SVD of matrix $A$
3. Discard all singular values below a threshold and the corresponding singular vectors. For $\nu$ singular values, this requires to store $\nu(m + n + 1)$ values. For small values of $\nu$, this is much smaller than $mn$. Store these values in a file.
4. Compute the approximate image $A_\nu = \sum_{j=1}^{\nu} \sigma_j u_j v_j^T$. This can be plotted. The approximation error in 2-norm is $\sigma_{\nu+1}$. 
Data analysis

- Assume you have data obtained by sampling an unknown function \( f(x) : \mathbb{R}^p \to \mathbb{R}^m \)
- \( n \) vectors of length \( m \) give matrix \( A \in \mathbb{R}^{m \times n} \)
- Idea: Do SVD of that matrix
- This will automatically detect the linear part of the function \( f \)
- Example: \( f(\alpha, \beta) = \alpha u_1 + \beta u_2 \) as unknown function
- The SVD of data generated by that function has 1 twice as a singular value, all others are 0
- In \( U \), you automatically (!) obtain as first two columns \( u_1 \) and \( u_2 \)
- This is at the core of all data analysis