Numerical Linear Algebra
Unit 8: Condition of a Problem

Numerical Analysis, Lund University

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A problem and its sensitivity

Let $X$, $Y$ be normed vector spaces and $f : X \rightarrow Y$ a (not necessarily linear) function.

We consider $f$ as a solution operator to a mathematical problem and call for $x^* \in X$ the task to compute $f(x^*)$ a problem for the in-data $x^*$.

**Definition**

$f(x^*)$ is called well-conditioned if $\|f(x^* + \delta x) - f(x^*)\|_Y$ is small for all small $\|\delta x\|_X$.

We quantify this more on the next slides.
Absolute condition number

Definition
Let $\delta f(\mathbf{x}^*) := f(\mathbf{x}^* + \delta \mathbf{x}) - f(\mathbf{x}^*)$. We call

$$\hat{\kappa}(\mathbf{x}^*) := \lim_{\delta \to 0} \sup_{\|\delta \mathbf{x}\|_X \leq \delta} \frac{\|\delta f\|_Y}{\|\delta \mathbf{x}\|_X}$$

the absolute condition number of the problem $f(\mathbf{x}^*)$.

Note: If $f$ is differentiable with a Jacobian $J$ we have

$$\hat{\kappa}(\mathbf{x}^*) = \|J(\mathbf{x}^*)\|_{X,Y}$$

(express $\delta f$ by a Taylor expansion of $f$ and neglect higher order terms.)
Relative condition number

Definition
Let $\delta f(x^*) := f(x^* + \delta x) - f(x^*)$. We call

$$\kappa(x^*) := \lim_{\delta \to 0} \sup_{\|\delta x\| \leq \delta} \frac{\|\delta f\|}{\|f\|} \frac{\|\delta x\|}{\|x\|}$$

the relative condition number of the problem $f(x^*)$.

Note: If $f$ is differentiable with a Jacobian $J$ we have

$$\kappa(x^*) = \frac{\|J(x^*)\|_{X,Y}}{\|f(x^*)\|/\|x^*\|}.$$ 

A problem is well-conditioned if $\kappa(x^*) \leq 10^6$ else it is ill-conditioned.

(This is a rule of thumb not a quantitative statement.)
Examples

1. A harmless example: $f(x) = \frac{x}{2}$. This has $\kappa(x) \equiv 1$.

2. $f(x) = x_1 - x_2$, $x \in \mathbb{R}^2$. The relative condition number is in $\infty$-norm

$$\kappa(x) = 2 \frac{|x_1 - x_2|}{\max(|x_1|, |x_2|)}$$

Note, that subtraction of nearly equal values is extremely sensitive with respect to perturbations.

3. Wilkinson’s example: Compute the roots $\xi_i = i$ of the polynomial

$$p(x) = \prod_{i=1}^{20} (x - i) = a_{20}x^{20} + \cdots a_1x + a_0$$
We note \( a_{15} = -1672280820 = -1.67 \ldots 10^9 \). We investigate (experimentally) how \( \xi_{15} = 15 \) is affected by changing \( a_{15} \) by 0.1 (relative change \( 0.6 \times 10^{-11} \)).

Experimentally we obtain
\[
(\xi_{15} + \delta \xi_{15}) = 15.6454330915 - 4.02899392111j
\]
which gives the estimate \( 6.8 \times 10^{10} \leq \kappa \).
In Python ....

```python
from sympy import *
from scipy import *
from scipy.linalg import eig

x = symbols('x')

def p(x):
    """
    Wilkinson polynomial
    """
    p=1
    for i in range(1,21):
        p*=(x-i)
    return p

p(1)  # a numeric value
p(x)  # a symbolic value
p(15)
pp=expand(p(x))
coeff=[pp.coeff(x,i) for i in range(0,21)]
```
Companion matrix

In Python ....

```python
def comp_matrix(coeff):
    n = len(array(coeff)) - 1
    c = zeros((n, n))
    c[:, -1] = -array(coeff[:-1])
    c = c + diag(ones((n - 1,)), -1)
    return c

c = comp_matrix(coeff)
print(’The roots of the polynomial \
’, sort(eig(c)[0]))
```

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We change now one coefficient ...

c15 = coeff[15]
coeff[15] += 0.1  # perturbation
rel_in_error = abs(0.1 / c15)
c = comp_matrix(coeff)
newroot15 = sort(eig(c)[0])[14]
rel_out_error = abs(newroot15 - 15)
condition_15 = rel_out_error / rel_in_error
Consider the problem $f : x \mapsto Ax$ (matrix–vector multiplication)

$$
\kappa = \lim_{\delta \to 0} \sup_{\parallel \delta x \parallel x \leq \delta} \frac{\parallel A(x+\delta x)-Ax \parallel}{\parallel Ax \parallel} = \lim_{\delta \to 0} \sup_{\parallel \delta x \parallel x \leq \delta} \frac{\parallel A\delta x \parallel}{\parallel \delta x \parallel} = \parallel A \parallel \frac{\parallel x \parallel}{\parallel Ax \parallel}
$$

In the special case that $A$ is invertible, there exists a $b$ with $x = A^{-1}b$ and consequently $\frac{\parallel x \parallel}{\parallel Ax \parallel} = \frac{\parallel A^{-1}b \parallel}{\parallel b \parallel} \leq \parallel A^{-1} \parallel$. Thus $\kappa \leq \parallel A \parallel \parallel A^{-1} \parallel$.

In the homework we will see that for some special $x$ we have equality, i.e. the estimate is sharp.
Solving a linear equation system

Consider the problem \( f: b \mapsto A^{-1}b \) (i.e. solving \( Ax = b \))

This is a matrix-vector multiplication problem with \( A^{-1} \). We get

\[ \kappa \leq \| A \| \| A^{-1} \| \]

by interchanging \( A \) with \( A^{-1} \) on the slide before.
Solving a linear equation system, $A$ as input

Consider the problem $f : A \mapsto A^{-1}b$ (i.e. solving $Ax = b$)
$A$ is now problem input and subject to perturbations $\delta A$

$$b = (A + \delta A)(x + \delta x) = Ax + A\delta x + \delta Ax + \delta A\delta x \approx 0$$

Thus, $A\delta x + \delta Ax = 0$ (asymptotically for small perturbations).

$$\Rightarrow \delta x = -A^{-1}(\delta A)x \Rightarrow \|\delta x\| \leq \|A^{-1}\|\|\delta A\|\|x\|$$
From this we get

\[
\frac{\|\delta x\|}{\|x\|} \leq \frac{\|A^{-1}\|\|\delta A\|}{\|\delta A\|} = \|A\|\|A^{-1}\| \leq \|A\|\|A^{-1}\|
\]

(The same estimate as before when we considered \(b\) as input.)
The number $\kappa_A = \|A\|\|A^{-1}\|$ is called the condition number of a matrix.

In the 2-norm it can be expressed by the singular values of $A$:

$$\kappa_A = \frac{\sigma_1}{\sigma_n}$$

where $\sigma_1$ is the largest and $\sigma_n$ the smallest singular value of $A$. 