Numerical Linear Algebra
Unit 8: Condition of a Problem

Numerical Analysis, Lund University

Claus Führer and Philipp Birken
A problem and its sensitivity

Let $X$, $Y$ be normed vector spaces and $f : X \rightarrow Y$ a (not necessarily linear) function.

We consider $f$ as a solution operator to a mathematical problem and call for $x^* \in X$ the task to compute $f(x^*)$ a problem for the *in-data* $x^*$. 

**Definition**  
$f(x^*)$ is called well-conditioned if $\|f(x^* + \delta x) - f(x^*)\|_Y$ is small for all small $\|\delta x\|_X$.

We quantify this more on the next slides.
**Absolute condition number**

**Definition**
Let \( \delta f(x^*) := f(x^* + \delta x) - f(x^*) \). We call

\[
\hat{\kappa}(x^*) := \lim_{\delta \to 0} \sup_{\delta \leq \delta x \leq 0} \frac{\|\delta f\|_Y}{\|\delta x\|_X}
\]

the absolute condition number of the problem \( f(x^*) \).

**Note:** If \( f \) is differentiable with a Jacobian \( J \) we have

\[
\hat{\kappa}(x^*) = \|J(x^*)\|_{X,Y}
\]

(express \( \delta f \) by a Taylor expansion of \( f \) and neglect higher order terms.)
Relative condition number

Definition
Let \( \delta f(x^*) := f(x^* + \delta x) - f(x^*) \). We call

\[
\kappa(x^*) := \lim_{\delta \to 0} \sup_{\|\delta x\| \leq \delta} \frac{\|\delta f\|}{\|f\| \|\delta x\| / \|x\|}
\]

the relative condition number of the problem \( f(x^*) \).

Note: If \( f \) is differentiable with a Jacobian \( J \) we have

\[
\kappa(x^*) = \frac{\|J(x^*)\|_{X,Y}}{\|f(x^*)\| / \|x^*\|}.
\]

A problem is well-conditioned if \( \kappa(x^*) \leq 10^6 \) else it is ill-conditioned.

(This is a rule of thumb not a quantitative statement.)
Examples

1. A harmless example: \( f(x) = \frac{x}{2} \). This has \( \kappa(x) \equiv 1 \).

2. \( f(x) = x_1 - x_2, \ x \in \mathbb{R}^2 \). The relative condition number is in \( \infty \)-norm

\[
\kappa(x) = \frac{2}{|x_1 - x_2|/ \max(|x_1|, |x_2|)}
\]

Note, that subtraction of nearly equal values is extremely sensitive with respect to perturbations.

3. Wilkinson’s example: Compute the roots \( \xi_i = i \) of the polynomial

\[
p(x) = \Pi_{i=1}^{20} (x - i) = a_{20}x^{20} + \cdots a_1x + a_0
\]
We note $a_{15} = -1672280820 = -1.67 \ldots 10^9$. We investigate (experimentally) how $\xi_{15} = 15$ is affected by changing $a_{15}$ by 0.1 (relative change $0.6 \times 10^{-11}$).

Experimentally we obtain
$$(\xi_{15} + \delta \xi_{15}) = 15.6454330915 - 4.02899392111j$$
which gives the estimate $6.8 \times 10^{10} \leq \kappa$
Wilkinson’s example (Cont.)

In Python ....

```python
from sympy import *
from scipy import *
from scipy.linalg import eig

x = symbols('x')

def p(x):
    """
    Wilkinson polynomial
    """
    p=1
    for i in range(1,21):
        p*=(x-i)
    return p

p(1)  # a numeric value
p(x)  # a symbolic value
p(15)

pp=expand(p(x))
coeff=[pp.coeff(x,i) for i in range(0,21)]
```
Companion matrix

In Python ....

```python
def comp_matrix(coeff):
    n = len(array(coeff)) - 1
    c = zeros((n, n))
    c[:, -1] = -array(coeff[:-1])
    c = c + diag(ones((n-1,)), -1)
    return c

c = comp_matrix(coeff)
print('The roots of the polynomial
', sort(eig(c)[0]))
```
Perturbing the problem

We change now one coefficient ...

```python
# Perturbation

coeff[15] += 0.1
rel_in_error = abs(0.1 / c15)
c = comp_matrix(coeff)
newroot15 = sort(eig(c)[0])[14]
rel_out_error = abs(newroot15 - 15)
condition_15 = rel_out_error / rel_in_error
```
Consider the problem \( f : x \mapsto Ax \) (matrix–vector multiplication)

\[
\kappa = \lim_{\delta \to 0} \sup_{\|\delta x\|_X \leq \delta} \frac{\|A(x + \delta x) - Ax\|_{Ax}}{\|Ax\|} = \lim_{\delta \to 0} \sup_{\|\delta x\|_X \leq \delta} \frac{\|A\delta x\|}{\|\delta x\|} = \|A\| \frac{\|x\|}{\|Ax\|}
\]

In the special case that \( A \) is invertible, there exists a \( b \) with \( x = A^{-1}b \) and consequently \( \frac{\|x\|}{\|Ax\|} = \frac{\|A^{-1}b\|}{\|b\|} \leq \|A^{-1}\| \). Thus \( \kappa \leq \|A\|\|A^{-1}\| \).

In the homework we will see that for some special \( x \) we have equality, i.e. the estimate is sharp.
Consider the problem $f : b \mapsto A^{-1}b$ (i.e. solving $Ax = b$)

This is a matrix-vector multiplication problem with $A^{-1}$. We get

$$\kappa \leq \|A\| \|A^{-1}\|$$

by interchanging $A$ with $A^{-1}$ on the slide before.
Solving a linear equation system, $A$ as input

Consider the problem $f : A \mapsto A^{-1}b$ (i.e. solving $Ax = b$)

$A$ is now problem input and subject to perturbations $\delta A$

$$b = (A + \delta A)(x + \delta x) = Ax + A\delta x + \delta Ax + \delta A\delta x = b$$

Thus, $A\delta x + \delta Ax = 0$ (asymptotically for small perturbations).

$$\Rightarrow \delta x = -A^{-1}(\delta A)x \Rightarrow \|\delta x\| \leq \|A^{-1}\| \|\delta A\| \|x\|$$
Solving a linear equation system, $A$ as input (Cont.)

From this we get

$$\frac{\|\delta x\|}{\|x\|} \leq \frac{\|A^{-1}\| \|\delta A\|}{\|\delta A\|} = \|A\| \|A^{-1}\|$$

(The same estimate as before when we considered $b$ as input.)
Condition Number of a Matrix

The number $\kappa_A = \|A\|\|A^{-1}\|$ is called the condition number of a matrix.

In the 2-norm it can be expressed by the singular values of $A$:

$$\kappa_A = \frac{\sigma_1}{\sigma_n}$$

where $\sigma_1$ is the largest and $\sigma_n$ the smallest singular value of $A$. 