In Brian Bradie: A Friendly Introduction to Numerical Analysis, the heat transfer in a cooling fin attached to an electronic device is described. The fin has circular cross section and is therefore called a “pin” fin.

The steady state temperature distribution $T(x)$ satisfies the two-point boundary value problem

$$\frac{d}{dx}\left(r(x)^2 \frac{d}{dx}T(x)\right) - \frac{2hr(x)}{k} \cdot (T(x) - T_A) = 0$$

with boundary conditions

$$T(0) = T_0, \quad -kT'(L) = h \cdot (T(L) - T_A).$$

Here $r(x)$ is the pin fin’s cross section at $x$; $h$ is the convection heat transfer coefficient; $k$ is the thermal heat conductivity of the fin; $T_A$ is the ambient temperature of the surrounding air; $T_0$ is the temperature of the device the fin is attached to; $L$ is the length of the fin.

The fin is made of steel (thermal conductivity $k = 14$ W/m·K), and is attached to a body of temperature $T_0 = 100$ C. The pin’s length is $L = 0.1$ m and its cross section for $x \in [0, L]$ is

$$r(x) = r_0 \left(1 - \frac{x}{2L}\right),$$

with $r_0 = 0.02$ m. The ambient air temperature is $T_A = 20$ C, and the coefficient $h = 20$ W/m²·K.

When all these data are inserted into the equation, the problem can be written

$$T'' + a(x)T' + b(x)T + c(x) = 0$$

with boundary conditions

$$T(0) = T_0; \quad hT(L) + kT'(L) = hT_A.$$ 

The task is to solve this problem on $[0, L]$, plot the solution on this interval, and specifically, to determine the temperature at the free tip, i.e., $T(L)$.

1) Determine expressions for the coefficients $a(x)$, $b(x)$ and $c(x)$. You will need them to program the discretization (9) below.
2) Introduce a grid on \([0, L]\) in such a way that you have \(N\) interior grid points, equally spaced, with

\[
x_0 = 0, \ldots, x_i = i\Delta x, \ldots, x_N = L - \Delta x/2, \quad x_{N+1} = L + \Delta x/2.
\] (6)

Here, the step size \(\Delta x\) is given by

\[
(N + 1)\Delta x = L + \Delta x/2,
\] (7)

or equivalently,

\[
\Delta x = \frac{L}{N + 1/2}.
\] (8)

\(N\) should be a parameter in your program so that you can easily affect the accuracy of your solution.

3) Approximate all derivatives with symmetric difference quotients and construct the linear system of equations that will approximate the differential equation. Let \(T_i\) denote the numerical approximation to \(T(x_i)\), and note that the vector with components \(T_i\) will be your numerical solution to the problem. Your system of equations will be, for \(i = 1, \ldots, N\),

\[
\frac{T_{i-1} - 2T_i + T_{i+1}}{\Delta x^2} + a(x_i)\frac{T_{i-1} - T_{i+1}}{2\Delta x} + b(x_i)T_i + c(x_i) = 0.
\] (9)

Collect terms and determine the coefficients in front of \(T_{i-1}\), \(T_i\) and \(T_{i+1}\) respectively, so that you can write (9) as a system of linear equations for the determination of \(T_1, \ldots, T_N\). You will need these coefficients when you create the matrix \(A\) in point 6) below.

4) The boundary values need special attention. At the left end point \((x = 0)\) we have the boundary condition \(T_0 = 100\), but the boundary condition at \(x = L\) is more difficult as it involves derivatives. With the special grid we have chosen in point 2) above, there is no grid point exactly at \(x = L\), so you will need to approximate

\[
T(L) \approx \frac{T_{N+1} + T_N}{2}.
\] (10)

Likewise, we approximate

\[
T'(L) \approx \frac{T_{N+1} - T_N}{\Delta x}.
\] (11)

Both approximations are 2nd order accurate. The boundary condition

\[
hT(L) + kT'(L) = hT_A
\] (12)

can then be approximated by

\[
h \frac{T_{N+1} + T_N}{2} + k \frac{T_{N+1} - T_N}{\Delta x} = hT_A.
\] (13)
Solve this equation for $T_{N+1}$ and eliminate this variable in the discretization (9). Note that this will imply some minor changes in the last equation of your equation system. Construct the system you have to program.

5) When you have constructed the system, you have to solve a linear system of equations,

\[ A \cdot T = y \]  \hspace{1cm} (14)

which you will solve with MATLAB’s backslash operator. After that, you can plot the solution vector $T$ versus the grid vector $x$. Don’t forget to plot $T$ all the way out to $x = L$! To find the temperature at the end point, you need to calculate the average (see eqn (10))

\[ T(L) \approx \frac{T_N + T_{N+1}}{2} \]

as these two approximations are at equal distance from the endpoint $x = L$, to the left and right of the endpoint, respectively.

6) When you program this problem in MATLAB you need to generate the $N \times N$ matrix $A$. If you have an $N - 1$ vector $\text{sup}$, an $N - 1$ vector $\text{sub}$ and an $N$ vector $\text{main}$, then you can construct an $N \times N$ tridiagonal matrix with subdiagonal $\text{sub}$, main diagonal $\text{main}$ and superdiagonal $\text{sup}$ by the command

\[ A = \text{diag(\text{sub},-1)} + \text{diag(\text{main},0)} + \text{diag(\text{sup},1)}; \]

Using this technique, all you need to do is to first generate the three vectors $\text{sup}$, $\text{main}$, $\text{sup}$, and install them in their right positions in the matrix. Make sure that the elements in the first and last row of $A$ are correct so that the boundary conditions are properly represented. Finally, you plot the solution vector $T$ versus the grid vector $x$ and compute the fin tip temperature. Don’t forget to plot $T$ all the way out to $x = L$!

7) If you have programmed everything correctly, it should be an easy task to find the temperature profile for pin fins of a different cross section. Can you find a cross section that gives a lower tip temperature than the one you found above? Describe the cross section function and plot the improved temperature profile. As an alternative, you could experiment with the heat transfer coefficients $h$ and $k$; this would indicate if there is a more suitable material for the cooling fin.