1. Finite difference approximation of derivatives
2. Finite difference methods for 2p-BVPs $\mathcal{L} u = f$
3. Newton’s method
4. Boundary conditions
5. Adaptive grids
6. Sturm–Liouville eigenvalue problems $\mathcal{L} u = \lambda u$
7. Toeplitz matrices
8. Convergence: The Lax Principle
1. Approximation of derivatives

First order approximations

**Forward difference**

\[ y'(x) = \frac{y(x + \Delta x) - y(x)}{\Delta x} + O(\Delta x) \]

**Backward difference**

\[ y'(x) = \frac{y(x) - y(x - \Delta x)}{\Delta x} + O(\Delta x) \]
Spatial symmetric approximation of derivatives

Second order approximations

**Symmetric difference quotients**

\[
y'(x) = \frac{y(x + \Delta x) - y(x - \Delta x)}{2\Delta x} + O(\Delta x^2)
\]

\[
y''(x) = \frac{y(x + \Delta x) - 2y(x) + y(x - \Delta x)}{\Delta x^2} + O(\Delta x^2)
\]
Matrix representation of forward difference

\[ y'(x) = \frac{y(x + \Delta x) - y(x)}{\Delta x} + O(\Delta x) \]

Introduce vectors \( y = \{y(x_i)\} \) and \( y' = \{y'(x_i)\} \)

\[
\begin{pmatrix}
  y_0' \\
  y_1' \\
  \vdots \\
  y_N'
\end{pmatrix}
\approx
\frac{1}{\Delta x}
\begin{pmatrix}
  -1 & 1 & & \\
  -1 & 1 & & \\
  \ddots & \ddots & \ddots & \\
  -1 & 1 & & \\
\end{pmatrix}
\begin{pmatrix}
  y_0 \\
  y_1 \\
  \vdots \\
  y_{N+1}
\end{pmatrix}
\]
Note  Forward difference $\sim (N + 1) \times (N + 2)$ matrix

$$
\begin{pmatrix}
y_0' \\
y_1' \\
\vdots \\
y_N'
\end{pmatrix} \approx \frac{1}{\Delta x}
\begin{pmatrix}
-1 & 1 & & & \\
-1 & -1 & 1 & & \\
& \ddots & \ddots & \ddots & \\
& & -1 & 1 & \\
\end{pmatrix}
\begin{pmatrix}
y_0 \\
y_1 \\
\vdots \\
y_N+1
\end{pmatrix}
$$

Nullspace spanned by $y = (1 \ 1 \ 1 \ldots \ 1)^T$

Compare nullspace of $\frac{d}{dx}$, $y = 1 \Rightarrow y' \equiv 0$

Analogous result for backward difference
From derivatives to matrices. . .

Central difference

\[ y'(x) \approx \frac{y(x + \Delta x) - y(x - \Delta x)}{2\Delta x} \]

Matrix representation

\[
\begin{pmatrix}
  y'_1 \\
  y'_2 \\
  \vdots \\
  y'_N
\end{pmatrix}
\approx
\frac{1}{2\Delta x}
\begin{pmatrix}
  -1 & 0 & 1 \\
  \vdots & \vdots & \vdots \\
  -1 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
  y_0 \\
  y_1 \\
  \vdots \\
  y_{N+1}
\end{pmatrix}
\]
Note \( N \times (N + 2) \) matrix

\[
\begin{pmatrix}
y'_1 \\
y'_2 \\
\vdots \\
y'_N
\end{pmatrix}
\approx \frac{1}{2\Delta x}
\begin{pmatrix}
-1 & 0 & 1 \\
\vdots & \vdots & \vdots \\
-1 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
y_0 \\
y_1 \\
\vdots \\
y_{N+1}
\end{pmatrix}
\]

Nullspace is now two-dimensional

\[
\tilde{y} = (1 \ 1 \ 1 \ldots \ 1)^T \quad \text{and} \quad \hat{y} = (1 \ -1 \ 1 \ -1 \ldots \ 1)^T
\]
"False" nullspace

\[ \hat{y} = (1 \ -1 \ 1 \ -1 \ldots 1)^T \] does not converge to a \( C^1 \) function!

Compare difference equation \( y_{n+1} - y_{n-1} = 0 \), with characteristic equation

\[ z^2 - 1 = 0 \quad \Rightarrow \quad z = \pm 1 \]

and two solutions \( \bar{y}_n = 1 \) and \( \tilde{y}_n = (-1)^n \)
2nd order derivatives → matrices

\[ y'' = \frac{d^2y}{dx^2} \]

**Central difference**

\[ y''(x) \approx \frac{y(x + \Delta x) - 2y(x) + y(x - \Delta x)}{\Delta x^2} \]

\[
\begin{pmatrix}
  y_1'' \\
  y_2'' \\
  \vdots \\
  y_N''
\end{pmatrix}
\approx \frac{1}{\Delta x^2}
\begin{pmatrix}
  1 & -2 & 1 \\
  \vdots & \vdots & \vdots \\
  1 & -2 & 1
\end{pmatrix}
\begin{pmatrix}
  y_0 \\
  y_1 \\
  \vdots \\
  y_{N+1}
\end{pmatrix}
\]

**Note**  \( N \times (N + 2) \) matrix with 2D nullspace spanned by

\[
\tilde{y} = (1 \ 1 \ldots 1)^T \quad \text{and} \quad \hat{y} = (0 \ 1 \ 2 \ 3 \ldots N + 1)^T
\]
Nullspace of \( \frac{d^2}{dx^2} \)

\[ y = 1 \text{ and } y = x \text{ both have } y'' \equiv 0 \]

Compare difference equation \( y_{n+1} - 2y_n + y_{n-1} = 0 \), with characteristic equation

\[
z^2 - 2z + 1 = 0 \quad \Rightarrow \quad z = 1, 1
\]

and two solutions \( \bar{y}_n = 1 \) and \( \hat{y}_n = n \), respectively

This corresponds directly to \( y = 1 \) and \( y = x \)
Consider simplest problem

\[ y'' = f(x, y) \]
\[ y(0) = \alpha; \quad y(1) = \beta \]

Introduce equidistant grid with \( \Delta x = 1/(N + 1) \)

**FDM discretization**

\[ \frac{y_{i+1} - 2y_i + y_{i-1}}{\Delta x^2} = f(x_i, y_i) \quad i = 1 : N \]

\[ y_0 = \alpha; \quad y_{N+1} = \beta \]
Discrete 2pBVP

Equation system $F(y) = 0$

$$
F_1(y) = \frac{\alpha - 2y_1 + y_2}{\Delta x^2} - f(x_1, y_1)
$$

$$
F_i(y) = \frac{y_{i-1} - 2y_i + y_{i+1}}{\Delta x^2} - f(x_i, y_i)
$$

$$
F_N(y) = \frac{y_{N-1} - 2y_N + \beta}{\Delta x^2} - f(x_N, y_N)
$$

A (nonlinear) system $F(y) = 0$ for $N$ unknowns $y_1, y_2, \ldots, y_N$

Note how boundary values enter
3. Newton’s method

Let $y^{(k)}$ approximate the solution $y$ and expand in Taylor series

$$0 = F(y) = F(y^{(k)} + y - y^{(k)}) \approx F(y^{(k)}) + F'(y^{(k)}) \cdot (y - y^{(k)})$$

Define $y^{(k+1)}$ by

$$0 =: F(y^{(k)}) + F'(y^{(k)}) \cdot (y^{(k+1)} - y^{(k)})$$

**Newton’s method** (mathematical formulation)

$$y^{(k+1)} := y^{(k)} - [F'(y^{(k)})]^{-1} F(y^{(k)})$$
Jacobian matrix \( F'(y) = \{ \partial F_i / \partial y_j \} \)

For the FDM the 2p-BVP Jacobian matrix is

\[
F'(y) = \text{tridiag}(1/\Delta x^2, -2/\Delta x^2 - \frac{\partial f}{\partial y_i}, 1/\Delta x^2)
\]

**Tridiagonal matrix**, with

- Super- and subdiagonal elements \( 1/\Delta x^2 \)
- Diagonal elements \( -2/\Delta x^2 - \frac{\partial f}{\partial y_i} \)
- Sparse LU decomposition runs in \( O(N) \) time
- Solution effort moderate even when \( N \) is large
Newton’s method for $F(y) = 0$

Newton iteration

1. Compute Jacobian $F'(y^{(k)}) = \{\partial F_i / \partial y_j\}$

2. Factorize Jacobian matrix $F'(y^{(k)}) \rightarrow LU$

3. Solve linear system $LU\delta y^{(k)} = -F(y^{(k)})$

4. Update $y^{(k+1)} := y^{(k)} + \delta y^{(k)}$

Newton’s method is quadratically convergent
Quadratic convergence

Newton’s method converges if

1. \( \| F'(y^{(k)})^{-1} \| \leq C' \)
2. \( \| F''(y^{(k)}) \| \leq C'' \)
3. \( \| y^{(0)} - y \| < \varepsilon \) (close enough starting value)

Then convergence is quadratic

\[
\| y^{(k+1)} - y \| \leq C \cdot \| y^{(k)} - y \|^2
\]
4. Boundary conditions come in many types

In many cases the problem is linear, but boundary conditions vary

- **Dirichlet conditions**
  \[ y(0) = \alpha; \quad y(1) = \beta \] straightforward to implement

- **Neumann conditions**
  \[ y'(0) = \gamma; \quad y(1) = \beta \] requires special attention

- **Robin conditions**
  \[ y(0) + c \cdot y'(0) = \kappa; \quad y(1) = \beta \] requires same attention

for the method’s convergence order to be preserved
Example

\[ y'' = f(x, y) \]
\[ y(0) = \alpha; \quad y'(1) = \beta \]

Equidistant grid, with \( x = 1 \) between grid points!

\[ x_N + \Delta x/2 = 1 = x_{N+1} - \Delta x/2 \]

\[ y'(1) = \beta \quad \Rightarrow \quad \frac{y_{N+1} - y_N}{\Delta x} = \beta \]

\[ \Rightarrow \quad y_{N+1} := \beta \Delta x + y_N \text{ is of second order at } x = 1 \]
Robin problem

Example

\[ y'' = f(x, y) \]
\[ y(0) = \alpha; \quad y(1) + cy'(1) = \kappa \]

Equidistant grid, with \( x = 1 \) between grid points

\[ x_N + \Delta x/2 = 1 = x_{N+1} - \Delta x/2 \]

\[ y(1) + cy'(1) = \kappa \quad \rightarrow \quad \frac{y_{N+1} + y_N}{2} + c \frac{y_{N+1} - y_N}{\Delta x} = \kappa \]

\[ \Rightarrow \quad y_{N+1} := \frac{(2c - \Delta x)y_N + 2\kappa\Delta x}{2c + \Delta x} \]
5. FDM on adaptive grids

Left and right divided differences

\[
D^- y_i = \frac{y_i - y_{i-1}}{x_i - x_{i-1}} = \frac{y_i - y_{i-1}}{h^-} \quad \quad D^+ y_i = \frac{y_{i+1} - y_i}{x_{i+1} - x_i} = \frac{y_{i+1} - y_i}{h^+}
\]

Then approximate derivatives by finite differences

\[
y_i' \approx \frac{h^- D^+ y_i + h^+ D^- y_i}{h^+ + h^-}
\]

\[
y_i'' \approx 2 \frac{D^+ y_i - D^- y_i}{h^+ + h^-}
\]

This is 2nd order only on smooth grids with \( h^+/h^- = 1 + O(N^{-1}) \)
Nonuniform grids

Grid deformation

A smooth nonuniform grid

![Graph showing a nonuniform grid with smooth deformation](image)
Adaptive grids

\[ u'' + 100u' = 100 \]
6. Sturm–Liouville eigenvalue problems

Diffusion problem

\[
\frac{\partial u}{\partial t} = \frac{\partial}{\partial x} \left( p(x) \frac{\partial u}{\partial x} \right) ; \quad u(t, a) = u(t, b) = 0
\]

Separation of variables (one space dimension)

\[
u(t, x) := y(x) \cdot v(t) \quad \Rightarrow \quad \frac{\dot{v}}{v} = \frac{(p(x) y')'}{y} =: \lambda
\]

Sturm–Liouville eigenvalue problem

\[
\frac{d}{dx} \left( p(x) \frac{dy}{dx} \right) = \lambda y \quad y(a) = 0 , \; y(b) = 0
\]
Sturm–Liouville eigenvalue problems...

Wave equation

\[ \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} ; \quad u(t, a) = u(t, b) = 0 \]

Express solution as \( u(t, x) = y(x) e^{i\omega t} \) \( \Rightarrow \)

\[ -\omega^2 y = c^2 y'' \quad y(a) = y(b) = 0 \]

Sturm–Liouville eigenvalue problem

\[ y'' = \lambda y \quad \text{with} \quad \lambda = -\omega^2 / c^2 \]
Why Sturm–Liouville eigenvalue problems?
Fluid–structure interaction

Tacoma Narrows Bridge 1940
von Kármán vortices

Fluid-structure interaction – mechanical resonance
Tuned mass dampers

Stockbridge damper (1926) – anti-fatigue devices
Compression loads, buckling and sun kinks
Music instruments

Metropolitan Opera Orchestra

Antoine Czurzyf

Paris
Taipei 101

91st Floor [390.60 m]
(Outdoor Observation Deck)

89th Floor [382.20 m]
(Indoor Observation Deck)

88th Floor

87th Floor
It rocks

Stay tuned – Mathematics rocks too!
Stationary Schrödinger equation

Particle in a box

Quantum mechanics

\[-\frac{\hbar^2}{2m} \frac{d^2\psi}{dx^2} + V(x)\psi = E\psi\]

This is a Sturm–Liouville eigenvalue problem, with energy levels $E_k$ defined by the eigenvalues
Sturm–Liouville eigenvalue problem

Find *eigenvalues* \( \lambda \) and *eigenfunctions* \( y(x) \) with

\[
\frac{d}{dx} \left( p(x) \frac{dy}{dx} \right) + q(x)y = \lambda y ; \quad y(a) = y(b) = 0
\]

**Discretization**  *Matrix eigenvalue problem*

\[
T_{\Delta x} y = \lambda_{\Delta x} y
\]

**Note**  Analytic eigenvalue problem converts to algebraic!
Consider \( y'' = \lambda y \) with boundary conditions \( y(0) = y(1) = 0 \)

Analytic solution

\[
y(x) = A \sin \sqrt{-\lambda} x + B \cos \sqrt{-\lambda} x
\]

Boundary values \( \Rightarrow B = 0 \) and \( A \sin \sqrt{-\lambda} = 0 \)

*Eigenvalues and eigenfunctions* for \( k = 1, 2, \ldots \)

\[
\lambda_k = -(k\pi)^2
\]

\[
y_k(x) = \sin k\pi x
\]

Fourier modes (harmonic analysis) associated with \( d^2/dx^2 \)
Discretization of $y'' = \lambda y$ with BVs $\Rightarrow$

$$\frac{y_{i-1} - 2y_i + y_{i+1}}{\Delta x^2} = \lambda_{\Delta x} y_i$$

$y_0 = y_{N+1} = 0$; $\Delta x = 1/(N + 1)$

Tridiagonal $N \times N$ matrix formulation

$$\frac{1}{\Delta x^2} \begin{pmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & \\ 1 & & & \ddots & \\ & \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{pmatrix} = \lambda_{\Delta x} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{pmatrix}$$
Discrete Sturm–Liouville problem...

Algebraic eigenvalue problem

\[ T_{\Delta x} y = \lambda_{\Delta x} y \]

Smallest eigenvalue

\[ \lambda_{\Delta x} = -\pi^2 + O(\Delta x^2) \]

The first few eigenvalues are well approximated, but the approximation gradually gets worse

**Note** There are only \( N \) discrete eigenvalues
Discrete Sturm–Liouville problem

First three eigenvectors of $T_{\Delta x}$ at $N = 19$
### Discrete Sturm–Liouville problem

<table>
<thead>
<tr>
<th></th>
<th>( \lambda_{\Delta x} )</th>
<th>(-9.8493)</th>
<th>(-39.1548)</th>
<th>(-87.1948)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exact eigenvalues ( \lambda )</td>
<td></td>
<td>(-9.8696)</td>
<td>(-39.4784)</td>
<td>(-88.8264)</td>
</tr>
<tr>
<td>Relative errors</td>
<td></td>
<td>0.21%</td>
<td>0.82%</td>
<td>1.84%</td>
</tr>
</tbody>
</table>

#### Note

- Lowest eigenvalues are more accurate
- Good approximations for \( \sqrt{N} \) first eigenvalues

(Here approximately first 4 – 5 modes)
Discrete Sturm–Liouville problem

High modes

Eigenvectors 7, 13, 19 of $T_{\Delta x}$ at $N = 19$
7. Toeplitz matrices

A *Toeplitz matrix* is constant along diagonals

**Example** (symmetric)

\[
T_{\Delta x} = \frac{1}{\Delta x^2} \begin{pmatrix}
-2 & 1 & 0 & \ldots \\
1 & -2 & 1 & \\
1 & -2 & 1 & \\
\ldots & \ldots & 0 & 1 & -2
\end{pmatrix}
\]
Much is known about Toeplitz matrices

- *Eigenvalues*
- *Norms*
- *Inverses*
- *etc.*

They can be generated in **MATLAB** using the built-in function \texttt{toeplitz}
Example  Solve the eigenvalue problem $Ty = \lambda y$ for

$$T = \begin{pmatrix}
-2 & 1 & 0 & \cdots \\
1 & -2 & 1 \\
1 & -2 & 1 \\
\vdots \\
\vdots & 0 & 1 & -2 \\
\end{pmatrix}$$

Note  $\lambda[T] = -2 + \lambda[S]$
Eigenvalues ... 

... the problem gets simplified

\[
Sy = \begin{pmatrix}
0 & 1 & 0 & \cdots \\
1 & 0 & 1 \\
1 & 0 & 1 \\
\vdots & 1 \\
\vdots & 1 \\
\vdots & 1 \\
\cdots & 1 \\
\cdots & 1 \\
\cdots & 1 \\
\end{pmatrix}
\begin{pmatrix}
y_1 \\
y_2 \\
\vdots \\
y_N \\
\end{pmatrix} = \lambda y
\]

Find eigenvalues \( \lambda[S] \), noting that the \( n^{\text{th}} \) equation of \( Sy = \lambda y \) is

\[y_{n+1} + y_{n-1} = \lambda y_n\]
Eigenvalues and difference equations

Linear difference equation \( y_{n+1} + y_{n-1} = \lambda y_n \) with boundary values \( y_0 = 0 = y_{N+1} \)

Characteristic equation \( z^2 - \lambda z + 1 = 0 \)

Two roots \( z \) and \( 1/z \) (product 1) implies general solution

\[ y_n = Az^n + Bz^{-n} \]

Boundary condition \( y_0 = 0 = A + B \) \( \Rightarrow \) \( y_n = A(z^n - z^{-n}) \)
Eigenvalues and difference equations. . .

Boundary condition \( y_{N+1} = 0 = A(z^{N+1} - z^{-(N+1)}) \) \( \Rightarrow \)

\[ z^{2(N+1)} = 1 \quad \Rightarrow \quad z_k = \exp\left(\frac{k\pi i}{N + 1}\right) \quad k = 1 : N \]

Sum of the roots of \( z^2 - \lambda z + 1 = 0 \) are

\[ \lambda_k[S] = z_k + 1/z_k = 2 \cos \frac{k\pi}{N + 1} \]

Hence

\[ \lambda_k[T] = -2 + 2 \cos \frac{k\pi}{N + 1} = -4 \sin^2 \frac{k\pi}{2(N + 1)} \]
Eigenvalue locations

\[
\lambda_k[T]
\]
Theorem  The $N \times N$ Toeplitz matrix

$$T = \begin{pmatrix} -2 & 1 & 0 & \ldots \\ 1 & -2 & 1 \\ 1 & -2 & 1 \\ \vdots \\ \ldots & 0 & 1 & -2 \end{pmatrix}$$

has $N$ real eigenvalues ($k = 1 : N$)

$$\lambda_k[T] = -4 \sin^2 \frac{k\pi}{2(N+1)} \in (-4, 0)$$
Eigenvalues of Toeplitz matrices

Consider $T_{\Delta x} := T/\Delta x^2$ with $\Delta x = 1/(N + 1)$ as an operator approximation

$$\frac{d^2}{dx^2} \leftrightarrow T_{\Delta x}$$

on $x \in [0, 1]$

**Corollary**  
*The eigenvalues of $T_{\Delta x}$ are*

$$\lambda_k[T_{\Delta x}] = -4(N + 1)^2 \sin^2 \frac{k\pi}{2(N + 1)} \approx -k^2\pi^2 = \lambda_k[d^2/dx^2]$$

for $k \ll N$
What are the norms of $T$?

**Lemma**  For a symmetric matrix $A$, it holds

$$\|A\|_2 = \max_k |\lambda_k|$$

**Lemma**  For a symmetric matrix $A$, it holds

$$\mu_2[A] = \max_k \lambda_k$$

(Both results actually hold for normal matrices)
Proofs. Norm

Definition

\[ \|A\|_2^2 = \max_{x^T x \neq 0} \frac{x^T A^T A x}{x^T x} \]

Find stationary points of the Rayleigh quotient of \( A^T A \), given by

\[ \rho(x) = \frac{x^T A^T A x}{x^T x} \]

\[ \text{grad}_x \rho(x) = \frac{(2A^T A xx^T x - 2xx^T A^T A x)}{(x^T x)^2} \implies \frac{A^T A x}{x^T x} = \rho(x)x \]

\[ A^T A x = \rho(x)x \implies A^2 x = \rho(x)x \]

So \( \rho(x) = \lambda^2 \), therefore \( \|A\|_2 = \max |\lambda[A]| \)
Proofs. Logarithmic norm

Definition

\[ \mu_2[A] = \max_{x^T x \neq 0} \frac{x^T Ax}{x^T x} \]

Find stationary points of the Rayleigh quotient of \( A \), given by \( \rho(x) = x^T Ax/x^T x \)

\[ \text{grad}_x \rho(x) = [(A + A^T)xx^T x - 2xx^T Ax]/(x^T x)^2 := 0 \]

\[ \frac{1}{2}(A + A^T)x = \rho(x)x \quad \Rightarrow \quad Ax = \rho(x)x \]

So \( \rho(x) = \lambda \), therefore \( \mu_2[A] = \max \lambda[A] \)
What are the norms of $T_{\Delta x}$?

Eigenvalues of $T_{\Delta x} = T/\Delta x^2$ are

$$
\lambda_k[T_{\Delta x}] = -4(N + 1)^2 \sin^2 \frac{k\pi}{2(N + 1)}
$$

So $\|T_{\Delta x}\|_2 = |\lambda_N|$ and $\mu_2[T_{\Delta x}] = \lambda_1$

Theorem  The Euclidean norms of $T_{\Delta x}$ are

$$
\|T_{\Delta x}\|_2 \approx \frac{4}{\Delta x^2} \quad \mu_2[T_{\Delta x}] \approx -\pi^2
$$
The norm of $T_{\Delta x}^{-1}$

Recall that $\mu[A] < 0 \Rightarrow \|A^{-1}\| \leq -1/\mu[A]$

Approximate $y'' = f(x)$ with $y(0) = y(1) = 0$ by

$$T_{\Delta x}u = q$$

Note $\mu_2[T_{\Delta x}] \approx -\pi^2$ implies the existence of a unique solution, as

$$\|T_{\Delta x}^{-1}\|_2 \approx \frac{1}{\pi^2}$$
What norms to use

Euclidean and RMS norms

The norm of a function is measured in the $L^2$ norm

$$\| u \|_{L^2}^2 = \int_0^1 u(x)^2 \, dx$$

A corresponding discrete function (vector) is then measured in the root mean square (RMS) norm

$$\| u \|_{\Delta x}^2 = \sum_{i=1}^{N} u(x_i)^2 \Delta x = \frac{1}{N+1} \sum_{i=1}^{N} u(x_i)^2 = \frac{1}{N+1} \| u \|_{2}^2$$

Note  For the operator norm,

$$\| T_{\Delta x}^{-1} \|_{\Delta x} \equiv \| T_{\Delta x}^{-1} \|_{2}$$
Simplest model problem (1D Poisson equation)

\[ y'' = f(x) \]
\[ y(0) = \alpha; \quad y(1) = \beta \]

Equidistant discretization

\[ \frac{y_{i+1} - 2y_i + y_{i-1}}{\Delta x^2} = f(x_i) \]
\[ y_0 = \alpha; \quad y_{N+1} = \beta \]
Insert exact continuous solution $y(x)$ into discretization

$$\frac{y(x_{i-1}) - 2y(x_i) + y(x_{i+1})}{\Delta x^2} = f(x_i, y(x_i)) - l(x_i)$$

Taylor expansion of local error, using $f(x_i) = y''(x_i)$

$$-l(x_i) = 2 \left( \frac{\Delta x^2}{4!} y^{(4)}(x_i) + \frac{\Delta x^4}{6!} y^{(6)}(x_i) + \ldots \right)$$

Only even powers of $\Delta x$ due to symmetry
**Definition**  
*The global error is defined*  
\[ e(x_i) = y_i - y(x_i) \]

**Convergence**  
Will show that \( e(x) \to 0 \) as \( \Delta x \to 0 \), or more specifically  
\[ e(x_i) = c_1 \Delta x^2 + c_2 \Delta x^4 + \ldots \]

Again only *even powers* due to *symmetry*
Consider the problem $y'' = f$ discretized by 2nd order FDM

$$T_{Δx}u = f$$

with $T_{Δx}$ tridiagonal. Then

**Numerical solution**  
$T_{Δx}u = f$

**Exact solution**  
$T_{Δx}y(x) = f(x) - l(x)$

**Error equation**  
$T_{Δx}e(x) = l(x)$

where $e(x) = u - y(x)$ is the *global error*
Solve $T_{\Delta x} u = f$ formally to get

**Numerically** \[ u = T_{\Delta x}^{-1} \cdot f \]

**Exact** \[ y(x) = T_{\Delta x}^{-1} \cdot (f - l(x)) \]

**Global error** \[ e(x) = T_{\Delta x}^{-1} \cdot l(x) \]

**Error bound** \[ \| e(x) \|_{\Delta x} \leq \| T_{\Delta x}^{-1} \|_2 \cdot \| l(x) \|_{\Delta x} \]
Convergence. . .

Recall

- $\mu_2[T_{\Delta x}] \approx -\pi^2 \Rightarrow \| T_{\Delta x}^{-1} \|_2 \lesssim 1/\pi^2$

- $\| e(x) \|_{\Delta x} \leq \| T_{\Delta x}^{-1} \|_2 \cdot \| l(x) \|_{\Delta x}$

- $\| l \|_{\Delta x} = \gamma_1 \Delta x^2 + \gamma_2 \Delta x^4 \ldots$

We therefore have

$$\| e \|_{\Delta x} \leq C \cdot \| l \|_{\Delta x} = c_1 \Delta x^2 + c_2 \Delta x^4 + \ldots$$

and we have convergence as $\Delta x \to 0$
The Lax Principle

Conclusion

Consistency  local error  \( l \to 0 \)  as  \( \Delta x \to 0 \)

Stability  \( \| T_{\Delta x}^{-1} \|_2 \leq C \)  as  \( \Delta x \to 0 \)

Convergence  global error  \( e \to 0 \)  as  \( \Delta x \to 0 \)

Theorem  (Lax Principle)

\[ \text{Consistency} + \text{Stability} \Rightarrow \text{Convergence} \]

“Fundamental theorem of numerical analysis”