1.12: Multistep Methods in General

The general form of a linear multistep method reads

$$\sum_{i=0}^{k} \alpha_{k-i} u_{n+1-i} - h_n \sum_{i=0}^{k} \beta_{k-i} f(t_{n+1-i}, u_{n+1-i}) = 0.$$ 

For starting a multistep method $k$: starting values $u_0, \ldots, u_{k-1}$ are required.
1.13: Global Error

The quantity of interest is the global error of the method at a given time point $t_n$

$$e_n := y(t_n) - u_n,$$

with $n = t_n/h$.

If for exact starting values $e_n = O(h)$, then the method is said to be convergent. More precisely, a method is convergent of order $p$, if

$$e_n = O(h^p).$$
1.14: Local Residual

To make a statement about the behavior of the global error, we have to introduce and study first the local residual:

**Definition.** Let \( y \) be a differentiable function, then the quantity

\[
l(y, t_n, h) := \sum_{i=0}^{k} \alpha_{k-i}y(t_{n-i}) - h \sum_{i=0}^{k} \beta_{k-i}y(t_{n-i})
\]

is called the local residual of the method.
1.15: Example

The local residual of the two-step implicit Adams method is defined by

\[
l(y, t + h, h) = y(t + h) - y(t) - h \left( \frac{5}{12} \dot{y}(t + h) + \frac{8}{12} \dot{y}(t) - \frac{1}{12} \dot{y}(t - h) \right).\]

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1.16: Example (Cont.)

Taylor expansion leads to

\[
\begin{align*}
l(y, t, h) &= h\dot{y}(t) + \frac{1}{2} h^2 \ddot{y}(t) + \frac{1}{6} h^3 y^{(3)}(t) + \frac{1}{24} h^4 y^{(4)}(t) + \ldots \\
&\quad - \frac{5}{12} h\dot{y}(t) - \frac{5}{12} h^2 \ddot{y}(t) - \frac{5}{24} h^3 y^{(3)}(t) - \frac{5}{72} h^4 y^{(4)}(t) + \ldots \\
&\quad - \frac{8}{12} h\dot{y}(t) \\
&\quad + \frac{1}{12} h\dot{y}(t) - \frac{1}{12} h^2 \ddot{y}(t) + \frac{1}{24} h^3 y^{(3)}(t) - \frac{1}{72} h^4 y^{(4)}(t) + \ldots \\
&= -\frac{1}{24} h^4 y^{(4)}(t) + \ldots
\end{align*}
\]

The implicit two step Adams method has the order of consistency 3.
1.17: Order of Consistency

Conditions for higher order of consistency are given by the following theorem:

**Theorem.** A linear multistep method has the order of consistency $p$ if the following $p + 1$ conditions on its coefficients are met:

\[
\sum_{i=0}^{k} \alpha_i = 0
\]
\[
\sum_{i=0}^{k} i \alpha_i - \beta_i = 0
\]
\[
\sum_{i=0}^{k} \frac{1}{j!} i^j \alpha_i - \frac{1}{(j-1)!} i^{j-1} \beta_i = 0 \quad \text{with} \quad j = 2, \ldots, p.
\]
1.18: Asymptotic Form of Local Residual

The local residual of a method with order of consistency $p$ takes the form

$$l(y, t, h) = c_{p+1} h^{p+1} y^{(p+1)}(t) + O(h^{p+2}).$$

Adams–Bashforth methods have order of consistency $k$, Adams–Moulton methods have order of consistency $k + 1$, and BDF methods have order of consistency $k$.
1.19: Global Error Propagation

Consider (for simplicity) the linear differential equation $\dot{y} = Ay$. The difference of

$$\sum_{i=0}^{k} \alpha_{k-i} y(t_{n-i}) - h_n \sum_{i=0}^{k} \beta_{k-i} Ay(t_{n-i}) = l(y, t_n, h)$$

and

$$\sum_{i=0}^{k} \alpha_{k-i} u_{n-i} - h_n \sum_{i=0}^{k} \beta_{k-i} Au_{n-i} = 0.$$ 

gives a recursion for the global error:

$$\sum_{i=0}^{k} \alpha_{k-i} e_{n-i} - h_n \sum_{i=0}^{k} \beta_{k-i} Ae_{n-i} = l(y, t_n, h).$$
By introducing the vector

\[ E_n := \begin{pmatrix} e_n \\ e_{n-1} \\ \vdots \\ e_{n-k+1} \end{pmatrix} \in \mathbb{R}^{kn_y} \]

this recursion formula can be written in one-step form as

\[ E_{n+1} = \Phi_n(h)E_n + M_n \]
1.21: Global Error Propagation (Cont.)

with

$$\Phi_n(h) := \begin{pmatrix} -A_k^{-1}A_{k-1} & -A_k^{-1}A_{k-2} & \cdots & -A_k^{-1}A_1 & -A_k^{-1}A_0 \\ I & 0 & \cdots & 0 & 0 \\ 0 & I & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I & 0 \end{pmatrix},$$

$$A_i := (\alpha_i I - h\beta_i A) \text{ and}$$

$$M_n := \begin{pmatrix} -A_k^{-1}l(y, t_n, h) \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$
From this formula we see how the global error of a multistep method is built up. There is in every step a (local) contribution $M_n$, which is of the size of the local residual. Therefore, a main task is to control the integration in such a way that this contribution is kept small. The effect of these local residuals on the global error is influenced by $\Phi_n(h)$. The local effects can be damped or amplified depending on the properties of the propagation matrix $\Phi_n(h)$. This leads to the discussion of the stability properties of the method and its relation to the stability of the problem.
1.23: Stability

The stability requirement is

$$\|\Phi(h)^n\| < C$$

with $C$ being independent of $n$, which is equivalent to

**All eigenvalues of $\Phi(h)$ are within the unit circle and those on its boundary are simple.**