Consider the BVP
\[
\begin{align*}
- u''(x) + u(x) &= f(x), & 0 < x < 1, \\
 u(0) &= u(1) = 0.
\end{align*}
\]

In order to have a classical solution, i.e., \( u \in C^2 [0,1] \), and to use finite differences we require \( f \in C[0,1] \).

However, it is common with applications where \( f \) is not continuous, e.g., localized heat production.

Idea: Try to reformulate the BVP in such a way that a less regular solution is needed.

Weak formulation

For a classical solution \( u \in C^2_0 [0,1] = \{ u \in C^2 [0,1] : u(0) = u(1) = 0 \} \), we have for every \( v \in C^1_0 [0,1] = \{ v \in C^1 [0,1] : -v' \} \) that
\[
\int_0^1 (-u''(x) + u(x)) v(x) \, dx =\left[ u'(x)v(x) \right]_0^1 \bigg|_{\text{integration by parts}} + \int_0^1 u'(x)v'(x) + u(x)v(x) \, dx
\]
\[
= \int_0^1 f(x)v(x) \, dx.
\]

Set \( B(u,v) = \int_0^1 u'(x)v'(x) + u(x)v(x) \, dx \) and \( l(v) = \int_0^1 f(x)v(x) \, dx \), then the weak formulation of the BVP would be:

Find the (weak) solution \( u \) such that
\[
B(u,v) = l(v) \quad \forall v \in C^1_0 [0,1].
\]
NB 1) In order to make sense of \( B(u,v) = \int_0^1 u'v' + uv \, dx \)
the solution \( u \) only need to be one-time differentiable
(under an integral) and the right-hand side can make sense
even if \( f \) is not continuous.

2) Classical solution \( u \Rightarrow \) weak solution \( u \), but weak \( \neq \) classical

**Idea for a numerical scheme**

Instead of testing with all \( v \) is in \( C^1_0[0,1] \), we could
test with all \( v \) is in a finite dimensional space \( S_0, l \)
which has suitable approximation properties.
The numerical scheme (also called a Galerkin scheme) is then:

Find \( u_{dx} \in S_0, l \) such that

\[
B(u_{dx}, v) = l(v) \quad \forall v \in S_0, l.
\]

**NB** The numerical approximation \( u_{dx} \) is now a function!

**Question** How do we compute \( u_{dx} \)?

Let \( \{ \phi_i \}_{i=1}^M \) be a basis in \( S_0, l \). Then

\[
u_{dx}(x) = \sum_{i=1}^M c_i \phi_i(x) \quad \text{and} \quad v(x) = \sum_{i=1}^M a_j \phi_j(x).
\]

\[
\Rightarrow B(u_{dx}, \phi_j) = \sum_{i=1}^M a_j B(u_{dx}, \phi_j) = \sum_{i=1}^M a_j l(\phi_j)
\]

\( B \) is linear \( \quad \uparrow \quad \) \( l \) is linear

for all \( v \in S_0, l \), i.e., all \( a_j \in \mathbb{R} \).
\[ B(u_{\Delta x}, \phi_j) = l(\phi_j) \quad \forall \ j = 1, \ldots, M. \]
\[ \Rightarrow \sum_{i=1}^{M} c_i B(\phi_i, \phi_j) = l(\phi_j) \quad \Rightarrow \]
\[ A \cdot c = \begin{pmatrix} B(\phi_1, \phi_1) & \cdots & B(\phi_M, \phi_1) \\ \vdots & & \vdots \\ B(\phi_1, \phi_M) & \cdots & B(\phi_M, \phi_M) \end{pmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_M \end{pmatrix} = \begin{pmatrix} l(\phi_1) \\ \vdots \\ l(\phi_M) \end{pmatrix} = b_{\Delta x} \]

Hence, we obtain a linear system with \( M \) unknowns \( c_i \) and the numerical approximation \( u_{\Delta x} \) is given by \( \sum_{i=1}^{M} c_i \phi_i(x) \).

**Question:** How do we choose \( S_{\Delta x} \) and \{\phi_i\}?  

**Finite elements (one possible choice)**

Introduce a spatial grid: 
\[ \begin{array}{cccccccccc}
X_0 & x_1 & x_2 & \ldots & x_M & X_{M+1} = 1 \\
\end{array} \]
\[ x_0 = 0 \]
where \( \Delta x = \frac{1}{M+1} \).

\[ S_{\Delta x} = \left\{ v \in C[0,1] : v(x) = s_i x + t_i \text{ for } x \in [x_{i-1}, x_i], \right. \]
and \( v(0) = v(1) = 0 \)  

**Example (of a \( v \in S_{\Delta x} \))**

For this choice of \( S_{\Delta x} \), the hat functions \( \{\phi_i\}_{i=1}^{M} \) is a basis, 

where 
\[ 1 \rightarrow \phi_1 \rightarrow \phi_2 \rightarrow \ldots \rightarrow \phi_{M-1} \rightarrow \phi_M \rightarrow x \]
Question: How can a given function \( v \in C^1(0,1) \) be approximated by a \( v_{0x} \in S_{0x} \)?

Answer: Linear interpolation:

\[
v(x) \approx \sum_{k=1}^{M} v(x_i) \phi_i(x) = v_{0x}(x) \in S_{0x}
\]

Example:

\[v(x)\quad \cdots \quad v_{0x}(x)\]

\[x_0=0 \quad x_1 \quad x_2 \quad x_3 \quad x_4=1 \quad (M=3)\]

NB: For the basis functions \( \{\phi_i\} \) we have

\[
u_{0x}(x_j) = \sum_{i=1}^{M} c_i \phi_i(x_j) = c_j \phi_j(x_j) = c_j
\]

\[\phi_i(x_j) = 0 \quad \text{if} \quad i \neq j\]

Hence, the \( c_i \)'s can be interpreted as the value of the numerical solution \( u_{0x} \) at the gridpoints \( x_0, \ldots, x_M \), i.e.,
After choosing \( s_{\Delta x} \) and \( \{ \phi_i \}_{i=1}^n \), we can construct our linear system \( A_{\Delta x} c = (B(\phi_i, \phi_j)) c = b_{\Delta x} \).

1) \( B(\phi_i, \phi_{i+k}) = 0 \) for \( k = 2 \) as

\[
A_{\Delta x} = \begin{pmatrix}
1 & & & \\
& & \ddots & \\
& & & 1 \\
0 & & & \\
& & & \\
\end{pmatrix}
\]

with \( B(\phi_i, \phi_{i+1}) \) terms on the under and upper diagonals and \( B(\phi_i, \phi_i) \) terms on the main diagonal.

2) \( B(\phi_i, \phi_{i+1}) = \int_0^1 \phi_i' \phi_{i+1}' + \phi_i \phi_{i+1} \, dx \)

\[
= \int_{x_i}^{x_{i+1}} \phi_i' \phi_{i+1}' \, dx + \int_{x_i}^{x_{i+1}} \phi_i \phi_{i+1} \, dx
\]

Here,

\[
\phi_i'(x) = \begin{cases} 
\frac{1}{\Delta x} & \text{in } x \in [x_{i-1}, x_i] \\
-\frac{1}{\Delta x} & \text{in } x \in [x_i, x_{i+1}] \\
0 & \text{elsewhere}
\end{cases}
\]

\[
\Rightarrow \int_{x_i}^{x_{i+1}} \phi_i' \phi_{i+1}' \, dx = \int_{x_i}^{x_{i+1}} \left(-\frac{1}{\Delta x} \frac{1}{\Delta x} \right) \, dx = -\frac{1}{\Delta x}
\]

Furthermore, \( \int_0^{x_{i+1}} \phi_i \phi_{i+1} \, dx = \frac{\Delta x}{6} \Rightarrow B(\phi_i, \phi_{i+1}) = -\frac{1}{\Delta x} + \frac{\Delta x}{6} \)

3) \( B(\phi_i, \phi_i) = \ldots = \frac{2}{\Delta x} + \frac{2\Delta x}{3} \). 

\[ A_{\Delta x} = \frac{1}{\Delta x} \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & -1 & 2 & \\ & & & -1 & 2 \end{pmatrix} + \frac{\Delta x}{6} \begin{pmatrix} 4 \\ 1 \\ 4 \\ 1 \\ 4 \end{pmatrix} \]

\[ = -T_{\Delta x} \]

and \( b = \left( \int_0^1 f(x) \phi_1(x) \, dx, \ldots, \int_0^1 f(x) \phi_M(x) \, dx \right)^T. \)

**NB.** The integral \( \int_0^1 f(x) \phi_j(x) \, dx \) may not be exactly computable. If we make a linear interpolation of \( f \) we have

\[ f(x) \approx \sum_{i=1}^M f(x_i) \phi_i(x) \]

and

\[ b \approx \frac{\Delta x}{6} \begin{pmatrix} 4 \\ 1 \\ 4 \\ 1 \\ 4 \end{pmatrix} \begin{pmatrix} f(x_1) \\ \vdots \\ f(x_M) \end{pmatrix} \]

(compare with the calculations on the previous page).

2) **Finite differences:**

\[ (-\frac{1}{\Delta x^2} T_{\Delta x} + I) U = \begin{pmatrix} f(x_1) \\ \vdots \\ f(x_M) \end{pmatrix} \]

**Finite elements:**

\[ (-\frac{1}{\Delta x^2} T_{\Delta x} + \Delta x M_{\Delta x}) C = \Delta x M_{\Delta x} \begin{pmatrix} f(x_1) \\ \vdots \\ f(x_M) \end{pmatrix} \]

where \( M_{\Delta x} = \frac{1}{6} \begin{pmatrix} 4 & -1 & & & \\ -1 & 4 & -1 & & \\ & -1 & 4 & -1 & \\ & & -1 & 4 & \\ & & & -1 & 4 \end{pmatrix} \).
Question: Can we do something similar for equations in 2 and 3 space dimensions?

Answer: Yes, "just" replace integration by parts with Divergence theorem.

Consider the d-dimensional equation

\[
\begin{cases}
-\Delta u + u = f & \text{in } \Omega \subset \mathbb{R}^d \\
u = 0 & \text{on } \partial \Omega \subset \mathbb{R}^{d-1}
\end{cases}
\]

If \( d = 3 \):

Nabla notation

Let \( \mathbf{v} : \mathbb{R}^d \rightarrow \mathbb{R} \) and \( \mathbf{w} : \mathbb{R}^d \rightarrow \mathbb{R}^d \)

\[\nabla \mathbf{v} = \left( \frac{\partial v}{\partial x_1}, \ldots, \frac{\partial v}{\partial x_d} \right) \quad \text{[Gradient]}\]

\[\nabla \cdot \mathbf{w} = \sum_{i=1}^{d} \frac{\partial w_i}{\partial x_i} \quad \text{[Divergence]}\]

\[\Delta \mathbf{v} = \nabla \cdot \nabla \mathbf{v} = \sum_{i=1}^{d} \frac{\partial^2 v}{\partial x_i^2} \quad \text{[Laplace operator]}\]

Divergence theorem:

\[\int_{\Omega} \nabla \cdot \mathbf{w} \, dx = \int_{\partial \Omega} \mathbf{w} \cdot \mathbf{n} \, ds\]

1. \( \nabla \cdot (\mathbf{w} \mathbf{v}) = \sum_{i=1}^{d} \frac{\partial}{\partial x_i} (w_i v) = \sum_{i=1}^{d} \left( \frac{\partial w_i}{\partial x_i} v + w_i \frac{\partial v}{\partial x_i} \right)\]
   
   \[= (\nabla \cdot \mathbf{w}) \mathbf{v} + \mathbf{w} \cdot \nabla \mathbf{v}\]
2) \[ \int \nabla (\omega v) \, dx = \int (\nabla \omega) v + \omega \cdot \nabla v \, dx = \int (\omega \cdot n) v \, ds \]

\[ \Rightarrow \int (\omega \cdot n) v \, ds = \int (\omega \cdot n) v \, ds - \int \omega \cdot \nabla v \, dx \]

(Green's formula)

3) Set \( \omega = -\nabla u \). Then

\[ \int - (\Delta u) v \, dx = - \int (\nabla u \cdot n) v \, ds + \int \nabla u \cdot \nabla v \, dx \]

Compare with integration by parts in 1D:

\[ \int v \Delta u \, dx = - [u' v]_{0}^{1} + \int_{0}^{1} u' v' \, dx. \]

A classical solution \( u \in C^{2}_{0}(\Omega) \) to the \( d \)-dimensional equation (\( \Delta \)) also satisfies, for every \( v \in C^{1}_{0}(\Omega) \),

\[ \int (\nabla u + u) v \, dx = - \int (\nabla u \cdot n) v \, ds + \int \nabla u \cdot \nabla v + uv \, dx \]

\[ = 0 \quad \text{as} \quad v = 0 \quad \text{on} \quad \partial \Omega \]

\[ = \int f v \, dx. \]

Set \( B(u,v) = \int \nabla u \cdot \nabla v + uv \, dx \) and \( L(v) = \int f v \, dx \),

then the weak formulation of the \( d \)-dim. equation reads

\[ \Rightarrow \text{see next page} \]
weak formulation Find u such that

\[ B(u, v) = L(v) \quad \forall v \in C^1_0[0,1] \]

NB 1) This is the same structure as for the one dimensional case (d = 1).

2) To make a proper theory of existence and convergence the space \( C^1_2(\Omega) \) should be replaced by the Sobolev space \( H^1_0(\Omega) \), but we save this for a continuation course...

Finite elements (d = 2)

Instead of a spatial grid \( \{x_1, \ldots, x_N\} \) we introduce a triangulation of \( \Omega \).

Example \( \Omega = \text{rectangle} \).

Here, we can choose

\[ S_{\Delta x} = \{ V \in C(\Omega) : V \text{ is an affine function over each triangle, and } V = 0 \text{ on } \partial \Omega \} \]

and the pyramid functions \( \{ \phi_i \} \) is a basis.

\[ \phi_i(x_1, x_2) \]