Part 1. Vector norms, matrix norms and logarithmic norms

- Vector norms
- Matrix norms
- Inner products
- The logarithmic norm
- Logarithmic norm properties
- Applications
1. Vector norms

**Definition** A vector norm \( \| \cdot \| : \mathbb{X} \rightarrow \mathbb{R} \) satisfies

1. \( \| u \| \geq 0 \); \( \| u \| = 0 \iff u = 0 \)
2. \( \| \alpha u \| = |\alpha| \cdot \| u \| \)
3. \( \| u \| - \| v \| \leq \| u \pm v \| \leq \| u \| + \| v \| \)

A norm generalizes the notion of *distance* between points
Vector norms

**Definition**  
The $l^p$ norms are defined  
\[ \|x\|_p = \left( \sum_{k=1}^{N} |x_k|^p \right)^{1/p} \]

**Graph of the unit circle in $\mathbb{R}^2$ for $l^p$ norms**

*Unit circles for $p = 1$, $p = 2$ (Euclidean norm), and $p = \infty$*
2. Matrix norms

**Definition** The operator norm associated with the vector norm $\| \cdot \|$ is defined by

$$
\| A \| = \sup_{x \neq 0} \frac{\| Ax \|}{\| x \|}
$$

For every vector norm there is a corresponding matrix norm.

**Note**

1. $\| Ax \| \leq \| A \| \cdot \| x \|$
2. $\| AB \| \leq \| A \| \cdot \| B \|$
## Vector and matrix norms

<table>
<thead>
<tr>
<th>Vector norm</th>
<th>Matrix norm</th>
</tr>
</thead>
<tbody>
<tr>
<td>$|x|_1 = \sum_i</td>
<td>x_i</td>
</tr>
<tr>
<td>$|x|_2 = \sqrt{\sum_i</td>
<td>x_i</td>
</tr>
<tr>
<td>$|x|_\infty = \max_i</td>
<td>x_i</td>
</tr>
</tbody>
</table>

**Definition**  The *spectral radius* of a matrix is defined by

$$\rho[A] = \max |\lambda[A]|$$
3. Inner products

**Definition** A bilinear form $\langle \cdot, \cdot \rangle : X \times X \to \mathbb{C}$ satisfying

1. $\langle u, u \rangle \geq 0$; $\langle u, u \rangle = 0 \iff u = 0$
2. $\langle u, v \rangle = \langle v, u \rangle$
3. $\langle u, \alpha v \rangle = \alpha \langle u, v \rangle$
4. $\langle u, v + w \rangle = \langle u, v \rangle + \langle u, w \rangle$

An inner product generates the Euclidean norm $\langle u, u \rangle = \|u\|^2$

An inner product generalizes the notion of scalar product
Inner products

**Examples**

1. Scalar product in $\mathbb{R}^m$: $\langle u, v \rangle = u^T v$
   with corresponding Euclidean vector norm $\|u\|_2^2 = \sum_{k=1}^{N} |u_k|^2$

2. General inner product in $\mathbb{C}^m$: $\langle u, v \rangle_G = u^H G v$ for any symmetric positive definite matrix $G$

3. Inner product in $L^2[0,1]$: $\langle u, v \rangle = \int_0^1 uv \, dx$
   with corresponding $L^2$ norm of functions $\|u\|_{L^2}^2 = \int_0^1 |u|^2 \, dx$
Inner products and operator norms

**Theorem**  
*Cauchy–Schwarz inequality*  
\[-\|u\| \cdot \|v\| \leq \langle u, v \rangle \leq \|u\| \cdot \|v\|\]

**Definition**  
The operator norm associated with \(\langle \cdot, \cdot \rangle\) is  
\[\|A\|^2 = \sup_{x \neq 0} \frac{\langle Au, Au \rangle}{\|u\|^2}\]

Hence \(\langle Au, Au \rangle \leq \|A\|^2 \|u\|^2\)
Interlude

The problem of stability

Classical stability for ODEs

\[ \dot{x} = Ax; \quad x(0) = x_0 \]

Characterization  Elementary stability conditions

- \( \text{Re} \lambda_k < 0 \) (\( \Leftrightarrow \) \( e^{tA} \to 0 \) as \( t \to \infty \))
- \( \|e^{tA}\| \leq C \) for all \( t \geq 0 \)
- \( \frac{d\|e^{tA}\|}{dt} \leq 0 \) (\( \Leftrightarrow \) \( \|e^{tA}\| \leq 1 \) for all \( t \geq 0 \))
Consider non-autonomous linear system

\[ \dot{x} = A(t)x; \quad x(0) = x_0 \]

*Stability is no longer characterized by eigenvalues*

Even with constant eigenvalues in the left half plane the system can be unstable (Petrowski & Hoppenstedt)

**Problem** Under what conditions does \( \|x(t)\| \) remain bounded as \( t \to \infty \)?
The logarithmic norm

Note that for an inner product,

\[
\frac{d\|x\|^2}{dt} = \frac{d\langle x, x \rangle}{dt} = 2\langle x, \dot{x} \rangle
\]

and that if \( \dot{x} = Ax \), then

\[
\langle x, \dot{x} \rangle = \langle x, Ax \rangle \leq \mu_2[A] \cdot \langle x, x \rangle
\]

**Definition**  The **logarithmic norm** is defined by

\[
\mu[A] = \sup_{x \neq 0} \frac{\langle x, Ax \rangle}{\|x\|^2}
\]
4. The logarithmic norm $\mu[A]$

**Definition**  For general matrix norms the logarithmic norm is defined by

$$
\mu[A] = \lim_{h \to 0^+} \frac{\|I + hA\| - 1}{h}
$$

If $\dot{x} = Ax$, the following differential inequality holds

$$
d\|x\|/dt \leq \mu[A] \cdot \|x\|
$$

**Note**  The logarithmic norm may be *negative*. Solution bound

$$
\|x(t)\| \leq e^{t\mu[A]} \cdot \|x(0)\| ; \quad t \geq 0
$$
Why the logarithmic norm?

Crude estimate

\[
\frac{d\|x\|}{dt} \leq \|\dot{x}\| = \|Ax\| \leq \|A\| \cdot \|x\|
\]

Exponentially growing bound

\[
\|x(t)\| \leq e^{t\|A\|} \cdot \|x(0)\|
\]

Note  

*Because \( \mu[A] \leq \|A\| \) we always have*  

\[
e^{t\mu[A]} \leq e^{t\|A\|}
\]
Matrix and logarithmic norms

### Computation

<table>
<thead>
<tr>
<th>Vector norm</th>
<th>Matrix norm</th>
<th>Log norm $\mu[A]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$|x|_1 = \sum_i</td>
<td>x_i</td>
<td>$</td>
</tr>
<tr>
<td>$|x|_2 = \sqrt{\sum_i</td>
<td>x_i</td>
<td>^2}$</td>
</tr>
<tr>
<td>$|x|_\infty = \max_i</td>
<td>x_i</td>
<td>$</td>
</tr>
</tbody>
</table>

**Definition**  
*Spectral abscissa*  $\alpha[A] = \max \text{Re } \lambda[A]$
5. Logarithmic norm properties

**Theorem**  The logarithmic norm has the following basic properties, which hold for all matrices $A$ and $B$

1. $\mu[A] \leq \|A\|$
2. $\mu[A + zI] = \mu[A] + \text{Re } z$
3. $\mu[\alpha A] = \alpha \mu[A], \quad \alpha \geq 0$
4. $\mu[A + B] \leq \mu[A] + \mu[B]$
5. $\|e^{tA}\| \leq e^{t\mu[A]}, \quad t \geq 0$
Uniform Monotonicity Theorem

The condition $\mu[A] < 0$ is akin to $A$ being *negative definite*. Then $A$ has a bounded inverse. More precisely,

**Theorem** (Uniform Monotonicity Theorem) *If $\mu[A] < 0$ then $A$ is nonsingular and*

$$
\|A^{-1}\| \leq -1/\mu[A]
$$

**Proof** (Here only for the Euclidean norm) Note that $\forall x$

$$
x^T A x \leq \mu_2[A] \cdot x^T x
$$
Proof . . .

Suppose $\mu_2[A] < 0$. By the Cauchy-Schwarz inequality

$$-\|x\|_2 \cdot \|Ax\|_2 \leq x^T Ax \leq \mu_2[A] \cdot \|x\|_2^2 < 0$$

for all $x \neq 0$. Hence $Ax \neq 0$, so $A^{-1}$ exists! Put $x = A^{-1}y$ and rearrange to get

$$-\|y\|_2 \leq \mu_2[A] \cdot \|A^{-1}y\|_2 \quad \Rightarrow \quad \frac{\|A^{-1}y\|_2}{\|y\|_2} \leq -\frac{1}{\mu_2[A]}$$

Take maximum over $y$ to see that $\|A^{-1}\|_2 \leq -1/\mu_2[A]$ \qed
Consider Explicit Euler for $\dot{x} = Ax + f(t)$

$$
x_{n+1} = x_n + hAx_n + hf(t_n)
$$

$$
x(t_{n+1}) = x(t_n) + hAx(t_n) + hf(t_n) - h^2r_n
$$

Global error $e_n = x_n - x(t_n)$

$$
e_{n+1} = e_n + hAe_n + h^2r_n \implies
$$

$$
\|e_{n+1}\| \leq \|e_n\| + h\|A\| \cdot \|e_n\| + \|h^2r_n\|
$$
Recall

**Lemma** If \( u_{n+1} \leq (1 + h\mu)u_n + ch^2 \) with \( u_0 = 0 \), then

\[
    u_n \leq \frac{ch}{\mu}[ (1 + h\mu)^n - 1] \leq ch \frac{e^{\mu t_n} - 1}{\mu}
\]

if \( h\mu \geq 0 \). In case \( -1 < h\mu < 0 \), we have

\[
    \max_n u_n \leq -\frac{ch}{\mu}
\]
Classical convergence analysis . . .

\[ \| e_{n+1} \| \leq (1 + h\|A\|) \cdot \| e_n \| + h^2 r_n \]

The lemma applies with \( \mu = \|A\| \) and \( c = \max_n \|r_n\| \)

\[ \| e_n \| \leq h \max_n \|r_n\| \frac{e^{\|A\|t_n} - 1}{\|A\|} \]

“Convergence,” but exponentially growing bound

(Hopeless!)
Modern convergence analysis

Explicit Euler for $\dot{x} = Ax + f(t)$

$$x_{n+1} = x_n + hAx_n + hf(t_n)$$

$$x(t_{n+1}) = x(t_n) + hAx(t_n) + hf(t_n) - h^2 r_n$$

Global error $e_n = x_n - x(t_n)$

$$e_{n+1} = e_n + hAe_n + h^2 r_n \quad \Rightarrow$$

$$\|e_{n+1}\| \leq (\|I + hA\|) \cdot \|e_n\| + \|h^2 r_n\|$$
Modern convergence analysis . . .

Note that, using the log norm,

$$\mu[A] = \lim_{h \to 0^+} \frac{\|I + hA\| - 1}{h}$$

we have

$$\|I + hA\| = 1 + h\mu[A] + O(h^2)$$

as $h\|A\| \to 0$
Now the lemma applies with $\mu \approx \mu[A]$ and $c = \max_n ||r_n||$

$$||e_n|| \lesssim h \max_n ||r_n|| \frac{e^{\mu[A]t_n} - 1}{\mu[A]} ; \quad \mu[A] \geq 0$$

and in case $-1 < h\mu[A] < 0$ we have

$$\max_n ||e_n|| \lesssim -\frac{h \max_n ||r_n||}{\mu[A]}$$

Convergence, with "realistic" error bound, as $\mu[A] \leq ||A||$
Consider Implicit Euler for $\dot{x} = Ax + f(t)$

$$x_{n+1} = x_n + hAx_{n+1} + hf(t_{n+1})$$

$$x(t_{n+1}) = x(t_n) + hAx(t_{n+1}) + hf(t_{n+1}) - h^2r_n$$

When is it possible to solve $(I - hA)x_{n+1} = x_n + hf(t_{n+1})$?

Uniform monotonicity theorem guarantees unique solution if

$$\mu[hA - I] < 0 \iff \mu[hA] < 1$$

Easily satisfied, even without bound on $\|hA\|$
Part 2. Interpolation

- Polynomial interpolation
- Lagrange interpolation
- Basis functions
- Numerical integration
1. What is interpolation?

Interpolation is “the opposite” of discretization

**Problem**  Given a discrete grid function (a vector) $F = \{f_j\}_0^N$ defined on a grid $\{x_j\}_0^N$, find a continuous function $f(x)$ with the *interpolating property* $f(x_j) = f_j$

Compare digital–to–analog conversion

Typically the function $f$ *is sought among polynomials* or among trigonometric functions (Fourier analysis)
Naïve polynomial interpolation

\[ P_n(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n \]

\( n + 1 \) coefficients, \( n + 1 \) interpolation conditions \( P_n(x_j) = f_j \)

\[
\begin{pmatrix}
1 & x_0 & x_0^2 & \cdots & x_0^n \\
1 & x_1 & x_1^2 & \cdots & x_1^n \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & x_n & x_n^2 & \cdots & x_n^n \\
\end{pmatrix}
\begin{pmatrix}
c_0 \\
c_1 \\
\vdots \\
c_n \\
\end{pmatrix}
=
\begin{pmatrix}
f_0 \\
f_1 \\
\vdots \\
f_n \\
\end{pmatrix}
\]

Vandermonde matrix, nonsingular if \( x_i \neq x_j \), unique solution

Tedious and often ill-conditioned approach
2. Lagrange interpolation

On a grid \( \{x_0, x_1, \ldots, x_k\} \) construct a degree \( k \) polynomial basis

\[
\{\varphi_i(x)\}_{i=0}^{k}
\]

such that \( \varphi_i(x_j) = \delta_{ij} \) (the Kronecker delta)

**Theorem**  If the values \( f_j = f(x_j) \) are known for the function \( f(x) \), then the degree \( k \) polynomial

\[
P(x) = \sum_{j=0}^{k} \varphi_j(x) f_j
\]

interpolates \( f(x) \) on the grid:

\[
P(x_j) = f_j \quad \text{with} \quad P(x) \approx f(x) \quad \text{for all} \quad x
3. Basis functions

2nd degree Lagrange

$$\varphi_1(x) = \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)}$$
Table of Lagrange basis polynomials

\[ P_2(x) = f_0 \varphi_0(x) + f_1 \varphi_1(x) + f_2 \varphi_2(x) \]

\[
\begin{array}{c|cccccc}
  x & \varphi_0 & \varphi_1 & \varphi_2 & f & P_2 \\
  \hline
  x_0 & 1 & 0 & 0 & f_0 & f_0 \\
  x_1 & 0 & 1 & 0 & f_1 & f_1 \\
  x_2 & 0 & 0 & 1 & f_2 & f_2 \\
\end{array}
\]

\[
P_2(x) = f_0 \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} + f_1 \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} + f_2 \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)}
\]
4. Numerical integration

Numerical integration is the approximation of definite integrals

\[ I(f) = \int_{a}^{b} f(x) \, dx \]

**Problem** For many functions *no primitive function is known*. The integral cannot be calculated analytically

**Note** For *polynomials* an integral \( \int_{a}^{b} P(x) \, dx \) can always be computed analytically

**Idea** Approximate \( f(x) \approx P(x) \) and compute \( \int_{a}^{b} P(x) \, dx \)
Numerical integration...

Approximate \( f \approx P \) and substitute “infinite sum” by a finite sum

The integrand is sampled at a finite number of points

\[
I(f) = \sum_{i=1}^{n} w_i f(x_i) + R_n
\]

Here \( R_n = I(f) - \sum_{i=0}^{n} w_i f(x_i) \) is the integration error

Numerical integration method

\[
I(f) \approx \sum_{i=0}^{n} w_i f(x_i)
\]
Numerical integration...

Approximate using Lagrange 2nd degree interpolant

\[
\int_{a}^{b} f(x) \, dx \approx \int_{a}^{b} \sum_{i=0}^{2} f(x_i) \varphi_i(x) \, dx = \sum_{i=0}^{2} f(x_i) \int_{a}^{b} \varphi_i(x) \, dx
\]

Weights \( w_i = \int_{a}^{b} \varphi_i(x) \, dx \) can be computed once and for all

Numerical integration method

\[
\int_{a}^{b} f(x) \, dx \approx \sum_{i=0}^{2} w_i f(x_i)
\]
Part 3. Nonlinear equations

- Solving nonlinear equations
- Fixed points
- Newton’s method
- Application. Newton vs Fixed point in Implicit Euler
1. Solving nonlinear equations

We can have a single equation

\[ x - \cos x = 0 \]

but in general we have systems

\[ 4x^2 - y^2 = 0 \]
\[ 4xy^2 - x = 1 \]

Nonlinear equations may have

- no solution
- one solution
- any finite number of solutions
- infinitely many solutions
Nonlinear equations are solved by iteration, computing a sequence \( \{x^[k]\} \) of approximations to the root \( x^* \).

Definition  
**The error** is defined by \( e^[k] = x^[k] - x^* \).

Definition  
**The method converges if** \( \lim_{k \to \infty} ||e^[k]|| = 0 \).

Definition  
**The convergence is**
- **linear** if \( ||e^[k+1]|| \leq c \cdot ||e^[k]|| \) with \( 0 < c < 1 \);
- **quadratic** if \( ||e^[k+1]|| \leq c \cdot ||e^[k]||^p \) with \( p = 2 \);
- **superlinear** if \( p > 1 \);
- **cubic** if \( p = 3 \), etc.
2. Fixed points

**Definition**  \( x \) is called a *fixed point* of the function \( g \) if

\[
x = g(x)
\]

**Definition**  A function \( g \) is called *contractive* if

\[
\|g(x) - g(y)\| \leq L[g] \cdot \|x - y\|
\]

with  \( L[g] < 1 \) for all \( x, y \) in the domain of \( g \)

A contraction map reduces the distance between points
**Theorem**  Assume that $g$ is Lipschitz continuous on the compact interval $I$. Further,

- If $g : I \rightarrow I$ there exists an $x^* \in I$ such that $x^* = g(x^*)$

- If in addition $L[g] < 1$ on $I$, then $x^*$ is unique, and

$$x_{n+1} = g(x_n)$$

converges to the fixed point $x^*$ for all $x_0 \in I$

**Note**  Both conditions are absolutely essential!
Fixed Point Theorem

Existence and uniqueness

- **Left**  No condition satisfied – no $x^*$
- **Center** First condition satisfied – maybe multiple $x^*$
- **Right** Both conditions satisfied – unique $x^*$
Error bound in fixed point iteration

By the Lipschitz condition

\[ x^{[k+1]} - x^* = g(x^{[k]}) - g(x^*) \]
\[ = g(x^{[k]}) - g(x^{[k+1]}) + g(x^{[k+1]}) - g(x^*) \]

we have

\[ \| x^{[k+1]} - x^* \| \leq L \cdot \| x^{[k]} - x^{[k+1]} \| + L \cdot \| x^{[k+1]} - x^* \| \]

**Theorem**  If \( L[g] < 1 \), then the error in fixed point iteration is bounded by

\[ \| x^{[k+1]} - x^* \| \leq \frac{L[g]}{1 - L[g]} \| x^{[k]} - x^{[k+1]} \| \]
3. Newton’s method

Newton’s method solves $f(x) = 0$ using \textit{repeated linearizations}. Linearize at the point $(x^{[k]}, f(x^{[k]})$)
Newton’s method . . .

Straight line equation

\[ y - f(x^{[k]}) = f'(x^{[k]}) \cdot (x - x^{[k]}) \]

Define \( x = x^{[k+1]} \Rightarrow y = 0 \), so that

\[-f(x^{[k]}) = f'(x^{[k]}) \cdot (x^{[k+1]} - x^{[k]}) \]

Solve for \( x = x^{[k+1]} \), to get \textit{Newton’s method}

\[ x^{[k+1]} = x^{[k]} - \frac{f(x^{[k]})}{f'(x^{[k]})} \]
Expand $f(x^{[k+1]})$ in a Taylor series around $x^{[k]}$

$$f(x^{[k+1]}) = f(x^{[k]} + (x^{[k+1]} - x^{[k]}))$$

$$\approx f(x^{[k]}) + f'(x^{[k]}) \cdot (x^{[k+1]} - x^{[k]}) := 0$$

$$\Rightarrow$$

$$x^{[k+1]} = x^{[k]} - (f'(x^{[k]}))^{-1} f(x^{[k]})$$

**Definition**  
$f'(x^{[k]})$ is the *Jacobian matrix* of $f$, defined by

$$f'(x) = \left\{ \frac{\partial f_i}{\partial x_j} \right\}$$
Newton’s method . . .

Write Newton’s method as a fixed point iteration $x^{[k+1]} = g(x^{[k]})$ with iteration function

$$g(x) := x - \frac{f(x)}{f'(x)}$$

**Note** Newton’s method *converges fast if* $f'(x^*) \neq 0$, because $g'(x^*) = f(x^*)f''(x^*)/f'(x^*)^2 = 0$

Expand $g(x)$ in a Taylor series around $x^*$

$$g(x^{[k]}) - g(x^*) \approx g'(x^*)(x^{[k]} - x^*) + \frac{g''(x^*)}{2}(x^{[k]} - x^*)^2$$

$$x^{[k+1]} - x^* \approx \frac{g''(x^*)}{2}(x^{[k]} - x^*)^2$$
Newton’s method . . .

Convergence

Define the error by \( \varepsilon[k] = x[k] - x^* \), then

\[ \varepsilon[k+1] \sim (\varepsilon[k])^2 \]

Newton’s method is \textit{quadratically convergent}.

Fixed point iterations are typically only linearly convergent

\[ \varepsilon[k+1] \approx g'(x^*) \cdot \varepsilon[k] \]

A problem with Newton’s method is that \textit{starting values need to be close enough} to the root.
Convergence order and rate

**Definition**  
*The convergence order is* \( p \) *with (asymptotic) error constant* \( C_p \), *if*

\[
0 < \lim_{k \to \infty} \frac{\|\epsilon[k+1]\|}{\|\epsilon[k]\|^p} = C_p < \infty
\]

**Special cases**

\( p = 1 \)  
*Linear convergence*  
*Fixed point iteration*  
\( C_p = |g'(x^*)| \)

\( p = 2 \)  
*Quadratic convergence*  
*Newton iteration*  
\( C_p = \left| \frac{f''(x^*)}{2f'(x^*)} \right| \)
As \( y_{n+1} = y_n + hf(y_{n+1}) \) we need to solve an equation
\[
y = hf(y) + \psi
\]

**Note** All implicit methods lead to an equation of this form

**Theorem** *Fixed point iterations converge if* \( L[hf] < 1 \), *restricting the step size to* \( h < 1/L[f] \)

**Note** For stiff equations \( L[hf] \gg 1 \) so fixed point iterations will not converge; it is *necessary to use Newton’s method*