Review questions and study problems, week 4

1. True or false (justify your answer): Consider the Sturm-Liouville problem
\[ \frac{d}{dx} \left( (1 - 0.8 \sin^2 x) \frac{dy}{dx} \right) - \lambda y = 0, \quad y(0) = y(\pi) = 0. \]
The following discretization
\[ \frac{p_{n-1}y_{n-1} - 2p_n y_n + p_{n+1}y_{n+1}}{\Delta x^2} = \lambda \Delta x y_n, \quad n = 1 : N \]
where \( p_n = 1 - 0.8 \sin^2 \frac{n\pi}{N+1} \) is of order 2.

2. Give a \( 4 \times 4 \) example of
   - A tridiagonal symmetric Toeplitz matrix
   - A skew-symmetric Toeplitz matrix
   - A lower triangular Toeplitz matrix

3. Solve the linear difference equation
   \[ 6u_{j+2} - 5u_{j+1} + u_j = 0 \quad (j = 0 : N - 1) \]
   \[ u_0 = 1, \quad u_{N+1} = 0. \]

4. True or false (justify your answer): If \( \lambda[T] \) are the eigenvalues of \( T \) and \( \lambda[S] \) the eigenvalues of \( S \), then \( \lambda[T] + \lambda[S] \) are the eigenvalues of \( T + S \).

5. Let \( Au = \lambda u \). Show that \( A^{-1} \) has the eigenvalues \( 1/\lambda \).

6. True or false: If \( Au = \lambda u \), then \( e^{tA} \) has the eigenvalues \( e^{t\lambda} \).

7. In class we determined the eigenvalues of
   \[ T_{\Delta x} = \frac{1}{\Delta x^2} \text{tridiag}(1 \quad -2 \quad 1) \]
   (a) Sketch the location of the eigenvalues in the complex plane
(b) Sketch the location of the eigenvalues of $T_{\Delta x}^{-1}$. (Make sure that your sketches have some “reasonable scaling,” e.g. by indicating where the eigenvalues are in relation to the unit circle.)

(c) If $\Delta x \to 0$, where will the eigenvalues of $T_{\Delta x}^{-1}$ “cluster?”

8. Consider the initial value problem $\dot{u} = T_{\Delta x} u$ with initial condition $u(0) = v$.

(a) Give an upper bound for $\|e^{tT_{\Delta x}}\|_2$ for $t \geq 0$.

(b) Sketch the location of the eigenvalues of $e^{tT_{\Delta x}}$ in the complex plane. (You may consider time $t$ to be a fixed parameter.)

(c) Where do the eigenvalues of $e^{tT_{\Delta x}}$ “cluster” as $\Delta x \to 0$ for $t$ fixed?

(d) Where do they go as $t \to \infty$ for $\Delta x$ fixed?

(e) Can you give or suggest an upper bound for the inverse $\|e^{-tT_{\Delta x}}\|_2$ (where $t > 0$)? How does that inverse behave as $\Delta x \to 0$?

(f) Suppose we solve this initial value problem using the explicit Euler method. What condition on the time step $\Delta t$ is a minimum requirement for stability?

(g) Same question for the implicit Euler method.

(h) Which method is suitable when $\Delta x \to 0$?

9. Consider the diffusion equation $u_t = u_{xx}$ with homogeneous boundary conditions $u(t,0) = u(t,1) = 0$ and write

$$\frac{1}{2} \frac{d\|u\|_2^2}{dt} = \langle u, u_t \rangle$$

for the usual inner product $\langle v, u \rangle = \int v u \, dx$. Use integration by parts and Sobolev’s lemma (alternatively the logarithmic norm of $d^2/dx^2$) to show that

$$\|u(t, \cdot)\|_2 \leq e^{-t \pi^2/2} \|u(0, \cdot)\|_2$$

10. In class we determined the eigenvalues of symmetric Toeplitz matrix $T = \text{tridiag}(1 \ 0 \ 1)$ analytically.

(Difficult) Determine, with a similar methodology, the eigenvalues of the skew symmetric Toeplitz matrix $S = \text{tridiag}(-1 \ 0 \ 1)$.

11. Let $y'$ be approximated by the second order, symmetric difference quotient $S_{\Delta x} = S/(2\Delta x)$. What are the

(a) eigenvalues of $S_{\Delta x}$

(b) Euclidean norm of $S_{\Delta x}$
(c) Euclidean logarithmic norm of $S_{\Delta x}$

(Hint: In class we showed that the Euclidean norm is “sharp” for symmetric matrices, and those proofs can easily be modified to see that the Euclidean norm is also sharp for skew-symmetric matrices.)

12. Consider the 2pBVP $u'' + u' + u = f(x)$ with $u(0) = u(1) = 0$.

(a) Find (an upper bound of) the logarithmic norm of the operator

$$\frac{d^2}{dx^2} + \frac{d}{dx} + 1.$$

Does the problem have a unique solution for every right-hand side $f$? (Use the fact that $\mu[A+B] \leq \mu[A] + \mu[B]$ and combine it with the Uniform Monotonicity Theorem.)

(b) Introduce a suitable grid and discretize the equation above. Use the same techniques as in the previous problem to show that your discretization has a unique solution for every right-hand side $f$.

(c) Let $u_{\Delta x}$ denote the solution vector on the grid. Give a bound for $\|u_{\Delta x}\|_{\Delta x}$ in terms of $\|f\|_{\Delta x}$ and the logarithmic norm.

13. Consider the 2pBVP $y'' + \omega^2 y = g(x)$ with homogeneous boundary data $y(0) = y(1) = 0$.

(a) For what values of the parameter $\omega$ can you guarantee that there is a unique solution?

(b) Let $\omega = \pi$. What happens with the analytical solution? Why?