You should present your solution for this homework at the latest on 2017-11-27.

This assignment has 7 tasks.

All the functions must be properly documented and also tested. We recommend that you produce a report of your work with Jupyter-Notebook (see Lecture on Monday, Nov. 20). You may work and present in groups by two. Use the upload link http://www.maths.lth.se/na/courses/NUMA01/fiup/ to upload your code. Upload one file only, either a *.py-file or an IpythonNotebook file. You will get then an email later from one of the teaching assistants with a presentation time slot.

Quadrature

Theory

In this homework we will compute approximations to the integral

\[ I = \int_a^b f(x) \, dx. \]  

(1)

One method for doing this is by using the composite trapezoidal rule, given by the formula

\[ I_h = \frac{h}{2} (f(a) + f(b)) + h \sum_{i=1}^{n-1} f(x_i) \]  

(2)

where \( x_i = a + \frac{i}{n} (b - a), \) \( h = \frac{b-a}{n}. \)

Then \( I_h \approx I \) and the approximation gets better the smaller \( h \) is, i.e. the more points we divide the interval into. (Compare with the definition of the Riemann integral.)

Task 1

Write a function \( \text{ctrapezoidal}(f, a, b, n) \) which implements the trapezoidal approximation (2). Test this function for different \( n \) and compare your result to the exact integral. (Choose a simple function \( f \) that you can integrate by hand, e.g. \( e^x \). However, don’t make it too simple.)
Task 2

Write a program that calls \texttt{ctrapezoidal(f, a, b, n)} for an increasing number of discretization points \(n\) in a loop. Stop the loop when the difference of two successive results is less than a given tolerance and return the final approximation.

Task 3

Write a function that makes an accuracy plot of the type depicted in the following figure. This plots the error \(|I_h - I|\) against the step size \(h\).

We make a loglog-plot, because we expect the error to be proportional to \(h^2\) for small \(h\), that is, \(\text{error} = Ch^2\). Taking the logarithm of both sides yields

\[
\log \text{error} = 2 \log h + \log C.
\]

This means that we should see a straight line of slope 2 if everything is correct. You do not need to take all \(n = 1, 2, 3, \ldots\), it is enough to take e.g. \(n = 1, 2, 4, 8, \ldots\). (See the command \texttt{loglog} for making figures in a double logarithmic scale and \texttt{grid} for turning on the grid.)

Interpolation

Theory

We consider a method for interpolating a sequence of points, that is, finding a polynomial \(P\) (of lowest degree) such that \(P(x_i) = y_i\) for given points -often measurements- \((x_i, y_i)\). If \(N+1\) points are given, there is a unique polynomial of degree \(N\) which passes through these points.
How to find the interpolation polynomial?

Let

\[ P(x) = c_N x^N + c_{N-1} x^{N-1} + \cdots + c_1 x + c_0 \]

be a polynomial of degree \( N \) which interpolates the \( N + 1 \) points \( (x_i, y_i) \), i.e. \( P(x_i) = y_i \). For its coefficients we get the equation system

\[ P(x_i) = c_N x_i^N + c_{N-1} x_i^{N-1} + \cdots + c_1 x_i + c_0 = y_i, \quad i = 0, \ldots, N. \]

We write this on matrix form:

\[
\begin{pmatrix}
    x_0^N & x_0^{N-1} & \cdots & x_0^1 & x_0^0 \\
    x_1^N & x_1^{N-1} & \cdots & x_1^1 & x_1^0 \\
    \vdots & \vdots & \ddots & \vdots & \vdots \\
    x_N^N & x_N^{N-1} & \cdots & x_N^1 & x_N^0
\end{pmatrix}
\begin{pmatrix}
    c_N \\
    c_{N-1} \\
    \vdots \\
    c_0
\end{pmatrix}
= \begin{pmatrix}
    y_0 \\
    y_1 \\
    \vdots \\
    y_N
\end{pmatrix}
\tag{3}
\]

and denote the matrix \( V \), the coefficient vector \( c \) and the right-hand side \( y \), so that

\[ Vc = y. \]

Task 4

Given a vector \( x \) of length \( N + 1 \), write a Python function that constructs the matrix

\[
\begin{pmatrix}
    x_0^N & x_0^{N-1} & \cdots & x_0^1 & x_0^0 \\
    x_1^N & x_1^{N-1} & \cdots & x_1^1 & x_1^0 \\
    \vdots & \vdots & \ddots & \vdots & \vdots \\
    x_N^N & x_N^{N-1} & \cdots & x_N^1 & x_N^0
\end{pmatrix}
\]

where the \( x_i \) are the components of the vector \( x \) (numbered from zero). Recall the function `numpy.column_stack` to stack vectors horizontally.

Task 5

Write a function \texttt{interpoly} that computes the coefficient vector \( c \) according to (3), given the vectors \( x \) and \( y \). You can use `scipy.linalg.solve` to solve the equation system.

Task 6

Write a function \texttt{polyval}, which has \( c \) and \( z \) as input and which computes the polynomial

\[ P(z) = \sum_{i=0}^{N} c_i z^{N-i}. \]
Test these last three functions on the vectors

\[
\begin{align*}
x &= (0.0, 0.5, 1.0, 1.5, 2.0, 2.5) \\
y &= (-2.0, 0.5, -2.0, 1.0, -0.5, 1.0)
\end{align*}
\]

by plotting the polynomial \( P \) over \([0, 3]\). Plot also the points \((x_i, y_i)\) as small stars and make sure that the polynomial passes through these points.

Good luck!