Numerical Approximation
Slides and Course Notes

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0: Preamble

These notes in slide form are the skeleton of the course. They have to be completed by your own blackboard notes and comments to the exercises.

The notation and the structure follow a draft of a forthcoming the book on the subject by Armin Iske, Hamburg

1: Introduction

Definitions and notations to set-up the approximation problem.

1.1: Motivation

We consider a function

$$f : \Omega \to \mathbb{R}$$

in a set of functions $\mathcal{F}$, e.g. the set of all continuous functions.

Let $\mathcal{S} \subset \mathcal{F}$ be a subset of simpler functions,

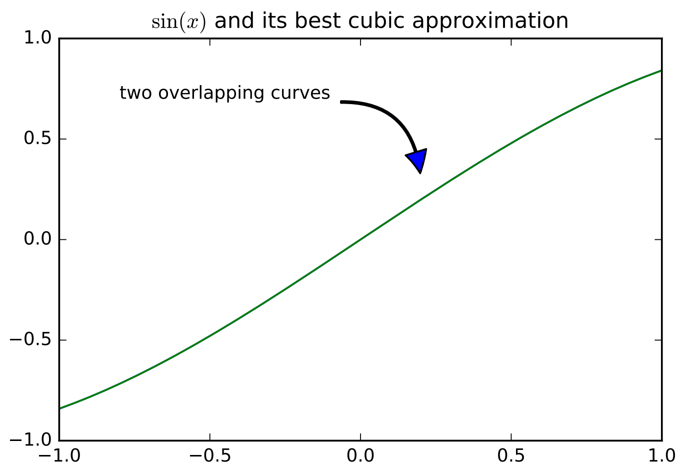
e.g polynomials of max degree 3.

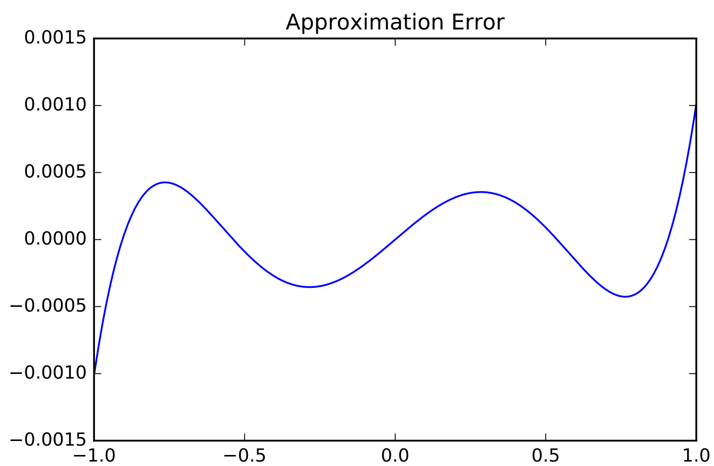
Then we ask us the questions,

- Is there a function s^* which approximates f best among all functions in $\mathcal{S}$?
- Is this function unique?

1.2: Example

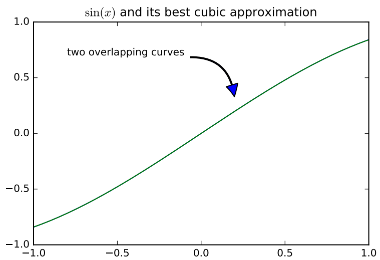
Let $\Omega = [-1, 1]$, $f = \sin$ and
let $\mathcal{S}$ be the set of all polynomials of (max) degree 3.

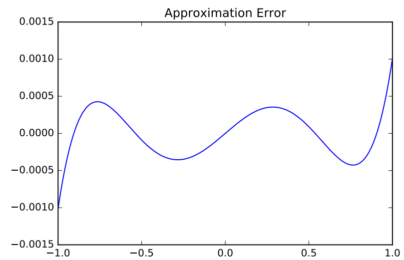




$$s^*(x) = -0.157615169549469\, x^3 + 0.998075138548952\, x$$

1.3: Example (cont.)





$$s^*(x) = -0.158\,x^3 + 0.998\,x$$

- How do we know that this is the best approximation?
- How do we measure the distance between two functions?
- How did we obtain this solution?

1.4: Normed linear spaces

We equip the set $\mathcal{F}$ with some structure, so that we have operations like addition, subtraction and scaling. Furthermore we ask for a tool to measure the size of its elements (or of the difference of two elements) which is essential for discussing the quality of approximations.

Definition.

$\mathcal{F}$ is called a $linear\ space$ over the field $\mathbb{R}$ if

1. $(\mathcal{F}, +)$ is a commutative group with respect to the addition

2. there is a scaling operation $\mathbb{R} \times \mathcal{F} \to \mathcal{F}$ such that

- $(\alpha \cdot \beta)f = \alpha(\beta f)$ *for all* $\alpha, \beta \in \mathbb{R}$ *and* $f \in \mathcal{F}$*,*
- $1f = f$ *for all* $f \in \mathcal{F}$*,*
- $\alpha(f + g) = \alpha f + \alpha g$ *for all* $\alpha \in \mathbb{R}$ *and* $f, g \in \mathcal{F}$*,*
- $(\alpha + \beta)f = \alpha f + \beta f$ *for all* $\alpha, \beta \in \mathbb{R}$ *and* $f \in \mathcal{F}$*.*

1.5: Normed linear spaces (Cont.)

The size of an element in a linear space is measured by a norm:

Definition.
Let $\mathcal{F}$ be a linear space. A map $\|\cdot\| : \mathcal{F} \to [0, \infty)$ is called a norm if

a) $\|f\| = 0 \Leftrightarrow f = 0$ *(definiteness)*

b) $\|\alpha f\| = |\alpha| \, \|f\|$ *(homogenity)*

c) $\|f + g\| \leq \|f\| + \|g\|$ *(triangular inequality)*

A linear space $(\mathcal{F}, \|\cdot\|)$ equipped with a norm is called a normed linear space.

1.6: Examples of normed linear spaces

Example.

- *Let* $\Omega \subset \mathbb{R}$ *be compact, then* $\mathcal{C}(\Omega) = \{u : \Omega \to \mathbb{R} : u \text{ continuous}\}$ *is a normed linear space with the maximum norm* $\|u\|_\infty := \max_{x\in\Omega} |u(x)|$*.*

- *Let* $\mathcal{P}^3$ *be the linear space of all cubic polynomials (max degree 3) and let* x_0, x_1, x_2, x_3 *be four distinct real values. Then*

$$\|p\| := \sum_{i=0}^{3} |p(x_i)|$$

  defines a norm on $\mathcal{P}^3$*.*

1.7: Norms and Convergence

Recall that the definition of convergence requires a norm:

Definition.
Let $(\mathcal{F}, \|\cdot\|)$ *be a normed linear space. A sequence* $\{f_i, i = 0, 1, 2 \ldots\} \subset \mathcal{F}$ *is called convergent, if there exits a* $f^* \in \mathcal{F}$ *such that for all* $\varepsilon > 0$ *exists an* $N \in \mathbb{N}$ *such that for all* $i > N : \|f_i - f^*\| < \varepsilon$*.*

Note, a sequence can be convergent with respect to one norm and nonconvergent with respect to another.

1.8: Cauchy Convergence and completeness

There is also a definition of convergence which does not make use of the limit f^*.

Definition.
Let $(\mathcal{F}, \|\cdot\|)$ *be a normed linear space. A sequence* $\{f_i, i = 0, 1, 2 \ldots\} \subset \mathcal{F}$ *is called Cauchy convergent, if*
for all $\varepsilon > 0$ *exists an* $N \in \mathbb{N}$ *s.t. for all* $i, j > N : \|f_i - f_j\| < \varepsilon$*.*

A normed linear space $\mathcal{F}$ is called complete if all Cauchy convergent sequences have a limit in $\mathcal{F}$.

Complete normed linear spaces are also called Banach spaces.

1.9: Example

Example. $(\mathcal{C}(\Omega), \|\cdot\|_\infty)$ *is a Banach space if* Ω *is a closed interval (compact).*

Proof:
Let $\{f_i, i = 0, 1, 2\ldots\}$ be a Cauchy sequence. Let $x \in \Omega$ be an arbitrary but fixed point, then $\{f_i(x), i = 0, 1, 2\ldots\}$ is a Cauchy sequence in $\mathbb{R}$. $\mathbb{R}$ is complete, thus $\exists y \in \mathbb{R}$ such that $\lim_{i\to\infty} f_i(x) = y$. This defines a function $f^*(x) := y$.
We show now, that f_i converges uniformly to f:
Let $\varepsilon > 0$ be given, then there is an N such that $\|f_i - f_j\|_\infty < \varepsilon \quad \forall i, j > N$.
Consequently,

$$|f^*(x) - f_j(x)| = \lim_{i\to\infty} |f_i(x) - f_j(x)| \leq \lim_{i\to\infty} \|f_i - f_j\|_\infty \leq \varepsilon$$

That means that f_i converges uniformly to f^*. As the uniform limit of a sequence of continuous functions is continuous $f^* \in \mathcal{C}(\Omega)$. $\square$

1.10: Euclidean spaces and Hilbert spaces

Example. *Let again $\Omega \subset \mathbb{R}$ be compact.*

- *For any $p \in \mathbb{N}$ is $\|f\|_p := (\int_\Omega |f(x)|^p \, \mathrm{dx})^{1/p}$ a norm on $\mathcal{C}(\Omega)$.*
- *In particular $\|f\|_2 := (\int_\Omega |f(x)|^2 \, \mathrm{dx})^{1/2}$ is the Euclidean norm on $\mathcal{C}(\Omega)$.*

The Euclidean norm can be defined by the inner product:

$(f, g)_2 = \int_\Omega f(x)g(x) \, \mathrm{dx}$ as $\|f\|_2 := \sqrt{(f, f)_2}$.

A normed linear space with an inner product norm is called an Euclidean space.

A complete Euclidean space is called a Hilbert space.

1.11: Smooth functions

We already met the space $\mathcal{C}(\Omega)$, the space of continuous functions.

We also consider

$$\mathcal{C}^k(\Omega) = \{u : \Omega \to \mathbb{R} \,:\, u \quad k-\text{times differentiable}\} \subset \mathcal{C}(\Omega)$$

This gives us a chain of spaces

$$\mathcal{C}^\infty(\Omega) \subset \cdots \subset \mathcal{C}^k(\Omega) \subset \mathcal{C}^{k-1}(\Omega) \subset \cdots \subset \mathcal{C}^0(\Omega) := \mathcal{C}(\Omega)$$

Here we defined the set of arbitrarily often differentiable functions by

$$\mathcal{C}^\infty(\Omega) := \bigcap_{k\in\mathbb{N}_0} \mathcal{C}^k(\Omega)$$

1.12: Finite dimensional subspaces

When approximating functions, we seek for a function $s \in \mathcal{S} \subset \mathcal{F}$ which describes the given function $f \in \mathcal{F}$ best in a sense we will define later.

If $\mathcal{S}$ is a finite dimensional subspace of $\mathcal{F}$ then its elements can be described by a finite amount of data - the coordinates with repect to a given basis:

$$s = \sum_{j=1}^{m} c_j s_j$$

Example: Polynomial space $\mathcal{P}^n = \left\{p : p(x) = \sum_{i=0}^{n} a_i x^i\right\}$. The basis functions are $\{1, x, x^2, \ldots, x^n\}$.

Note: $\mathcal{P}^n \subset \mathcal{C}^\infty(\Omega)$.

1.13: 2π-periodic functions

We also consider the space of 2π-periodic continuous functions:

$$\mathcal{C}_{2\pi} := \{u \in \mathcal{C}(\mathbb{R}) : u(x) = u(x + 2\pi)\}$$

It is a linear space (why?) and with $\|\cdot\|_p$ a normed space, with $\|\cdot\|_2$ an Eucledian space and with $\|\cdot\|$ a Banach space.