1.14: Trigonometric polynomials

**Definition.** A function $T : \mathbb{R} \to \mathbb{R}$ with

$$T(x) = \frac{a_0}{2} + \sum_{j=1}^{n} a_j \cos(jx) + b_j \sin(jx)$$

is called a trigonometric polynomial of degree $n$. The set of all trigonometric polynomials of degree $n$ is denoted by $\mathcal{T}^n$.

**Note:** $\mathcal{T}^n = \text{span}\{1, \sin(x), \cos(x), \cdots, \sin(nx), \cos(nx)\}$.

In signal processing one seeks for the best approximation of a function in $C_{2\pi}$ by a function in $\mathcal{T}^n$ with respect to $\| \cdot \|_2$. 
1.15 Fundamental questions

In approximation theory one studies the following questions

• Is there a best approximation?

• Is the best approximation unique?

• How is the (unique) best approximation characterized?

• Is there an algorithm for computing the best approximation?

• Can the approximation error be estimated?

• Is the approximation error converging to 0 by raising the dimension of $S$?
1.16 Error and convergence

We measure the approximation error by

\[ \eta(f, S) := \inf_{s \in S} \|s - f\| = \|s^* - f\| \]

and construct a chain of linear spaces with increasing dimensions

\[ S_0 \subset S_1 \subset \cdots \subset S_n \subset \mathcal{F} \]

in the hope of a convergence statement such as

\[ \lim_{n \to \infty} \eta(f, S_n) = 0 \]
1.17 Dense subsets

Definition.
The set $S := \bigcup_{n \geq 0} S_n$ is dense in $\mathcal{F}$ if

$$\lim_{n \to \infty} \eta(f, S_n) = 0$$
1.18 Two central theorems

We will prove later the two central theorems in this context:

**Theorem. Weierstrass**

The set of all polynomials $\mathcal{P} = \bigcup_{n \geq 0} \mathcal{P}_n$ is dense in $\mathcal{C}([a, b])$ with respect to the $\infty$-norm.

This implies

$$\forall f \in \mathcal{C}([a, b]) \ \forall \varepsilon > 0 \ \exists p \in \mathcal{P} \text{ such that } \|p - f\|_\infty < \varepsilon$$
1.19 Two central theorems (Cont.)

The next theorem states the speed of convergence of approximations with trigonometric polynomials.

**Theorem.** *Jackson*

*Let \( f \in C^k_{2\pi} \), then*

\[
\eta_{\infty}(f, T_n) \leq \left( \frac{\pi}{2(n + 1)} \right)^k \| f^{(k)} \|_{\infty} = O(n^{-k})
\]

The smoother the periodic function, the faster the convergence.