1.14: Trigonometric polynomials

Definition. *A function* $T : \mathbb{R} \to \mathbb{R}$ *with*

$$T(x) = \frac{a_0}{2} + \sum_{j=1}^{n} a_j \cos(jx) + b_j \sin(jx)$$

is called a trigonometric polynomial of degree n*. The set of all trigonometric polynomials of degree* n *is denoted by* $\mathcal{T}^n$*.*

Note: $\mathcal{T}^n = \mathrm{span}\{1, \sin(x), \cos(x), \cdots, \sin(nx), \cos(nx)\}$.

In signal processing one seeks for the best approximation of a function in $\mathcal{C}_{2\pi}$ by a function in $\mathcal{T}^n$ with respect to $\|\cdot\|_2$.

1.15 Fundamental questions

In approximation theory one studies the following questions

- Is there a best approximation?
- Is the best approximation unique?
- How is the (unique) best approximation characteized?
- Is there an algorithm for computing the best approximation?
- Can the approximation error be estimated?
- Is the approximation error converging to 0 by raising the dimension of S?

1.16 Error and convergence

We measure the approximation error by

$$\eta(f, \mathcal{S}) := \inf_{s \in \mathcal{S}} \|s - f\| = \|s^* - f\|$$

and construct a chain of linear spaces with increasing dimensions

$$\mathcal{S}_0 \subset \mathcal{S}_1 \subset \cdots \subset \mathcal{S}_n \subset \mathcal{F}$$

in the hope of a convergence statement such as

$$\lim_{n \to \infty} \eta(f, \mathcal{S}_n) = 0$$

1.17 Dense subsets

Definition.
The set $\mathcal{S} := \bigcup_{n\geq 0} \mathcal{S}_n$ *is dense in* $\mathcal{F}$ *if*

$$\lim_{n\to\infty} \eta(f, \mathcal{S}_n) = 0$$

1.18 Two central theorems

We will prove later the two central theorems in this context:

Theorem. *Weierstrass*
The set of all polynomials $\mathcal{P} = \bigcup_{n\geq 0} \mathcal{P}_n$ *is dense in* $\mathcal{C}([a,b])$ *with repect to the* ∞*-norm.*

This implies

$$\forall f \in \mathcal{C}([a,b]) \;\; \forall \varepsilon > 0 \;\; \exists p \in \mathcal{P} \text{ such that } \|p - f\|_\infty < \varepsilon$$

1.19 Two central theorems (Cont.)

The next theorem states the speed of convergence of approximations with trigonometric polynomials

Theorem. *Jackson*
Let $f \in \mathcal{C}^k_{2\pi}$*, then*

$$\eta_\infty(f, \mathcal{T}_n) \leq \left(\frac{\pi}{2(n+1)}\right)^k \|f^{(k)}\|_\infty = \mathcal{O}(n^{-k})$$

The smoother the periodic function, the faster the convergence.