2.9: Divided differences

Definition.
Let $p \in \mathcal{P}^n$ be the polynomial, which interpolates a given function $f \in \mathcal{C}[a,b]$ at the distinct points $\{x_i, i = 0, \dots, n\} \subset [a,b]$.
The leading coefficient of p is denoted by $f[x_0, x_1, \dots, x_n]$. It is called the divided difference of order n.

The name becomes clear from:

$$f(x) = \sum_{k=0}^{n} f(x_i) l_k(x) = \sum_{k=0}^{n} f(x_k) \prod_{\substack{i=0 \\ i \neq k}}^{n} \frac{x - x_i}{x_k - x_i} \Rightarrow$$

$$f[x_0, x_1, \dots, x_n] = \sum_{k=0}^{n} \frac{f(x_k)}{\Pi_{\substack{i=0 \\ i \neq k}}^{n} (x_k - x_i)}$$

2.10: Divided differences and higher derivatives

Theorem.
Let $f \in \mathcal{C}^n[a,b]$ *and let* $\{x_i, i = 0,\ldots,n\} \subset [a,b]$ *be ordered distinct points, i.e.* $x_i \leq x_{i+1}$.
Then there exists a point $\xi \in [x_0, x_n]$ *such that*

$$f[x_0, x_1, \ldots, x_n] = \frac{1}{n!} f^{(n)}(\xi)$$

For the proof we refer to Slide 2.??

2.11: Newton polynomials and interpolation

Theorem.
Let $p_n \in \mathcal{P}^n$ *be the polynomial which interpolates* f *at* $\{x_i, i = 0, \ldots, n\}$ *and let* $p_{n+1} \in \mathcal{P}^{n+1}$ *interpolate* f *at the same points and additionally at* $x_{n+1} \in [a, b]$.
Then

$$p_{n+1}(x) = p_n(x) + f[x_0, x_1, \ldots, x_{n+1}] \prod_{i=0}^{n} (x - x_i)$$

The polynomials $\omega_j(x) = \Pi_{i=0}^{j-1}(x - x_i) \in \mathcal{P}^j, j = 1, \ldots$ and $\omega_0(x) \equiv 1$ are called Newton polynomials.

2.12: Newton interpolation formula

A consequence of the theorem is the following representation of the interpolation polynomial:

$$p_n(x) = f[x_0] + f[x_0, x_1](x - x_0) + \cdots + f[x_0, x_1, \ldots, x_n] \prod_{i=0}^{n-1} (x - x_i)$$

with $f[x_0] := f(x_0)$.

2.13: Divided differences, recursion

$$
\begin{array}{c|ccccc}
 & k=0 & k=1 & k=2 & \dots & k=n \\
\hline
x_0 & f[x_0] & & & & \\
 & & f[x_0,x_1] & & & \\
x_1 & f[x_1] & & f[x_0,x_1,x_2] & & \\
 & & f[x_1,x_2] & & \ddots & \\
x_2 & f[x_2] & & \vdots & & f[x_0,\dots,x_n] \\
 & & \vdots & f[x_{n-2},x_{n-1},x_n] & & \\
\vdots & \vdots & & & & \\
 & & f[x_{n-1},x_n] & & & \\
x_n & f[x_n] & & & &
\end{array}
$$

2.14: Divided differences, recursion (Cont.)

Theorem.

$$f[x_j, \ldots, x_{j+k+1}] = \frac{f[x_{j+1}, \ldots, x_{j+k+1}] - f[x_j, \ldots, x_{j+k}]}{x_{j+k+1} - x_j}$$

For the proof consider a polynomial p_k which interpolates f at $x_j, \ldots, x_{j+k}$ and a polynomial q_k which interpolates at $x_{j+1}, \ldots, x_{j+k+1}$. Then discuss

$$p_{k+1}(x) = \frac{(x - x_j)q_k(x) - (x_{j+k+1} - x)p_k(x)}{x_{j+k+1} - x_j}$$

2.15: Hermite-Genochi Formula

Let us consider an n-simplex Δ_n:

$$\Delta_n := \left\{(\lambda_1, \ldots, \lambda_n) \in \mathbb{R} | \lambda_k \geq 0 \,\wedge\, \sum \lambda_k \leq 1\right\}$$

Example:

- $n = 1$: unit interval $[0, 1]$.
- $n = 2$: unit triangle.
- $n = 3$: unit tetrahedron.

2.16: Hermite-Genochi Formula (Cont.)

Theorem.

$$f[x_0,\dots,x_n] = \int_{\Delta_n} f^{(n)}(x_0 + \sum_{k=1}^{n} \lambda_k (x_k - x_0))\mathrm{d}\lambda$$

Let us exemplify this at $n = 1$:

$$f[x_0,x_1] = \int_0^1 f'(x_0+\lambda_1(x_1-x_0))\mathrm{d}\lambda = \frac{1}{x_1 - x_0}\int_{x_0}^{x_1} f'(\xi)\mathrm{d}\xi = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

This serves also as the base case of an induction proof of the above theorem.