2.17: Another Relation to Higher Derivatives

Theorem.
Let $f \in \mathcal{C}^n[a,b]$ *and let* $x_{\min} = \min_i x_i$ *and* $x_{\max} = \max_i x_i$*.*
Then there exists a $\xi \in [x_{\min}, x_{\max}]$ *with*

$$f[x_0, \ldots, x_n] = \frac{f^{(n)}(\xi)}{n!}$$

This theorem is proved by noting that the error function $g = f - p$ has at least $n + 1$ zeros, where $p \in \mathcal{P}^n$ is the polynomial which interpolates f at the x_i. By Rolle's theorem g' has at least n zeros and finally $g(n)$ has at least one zero in the interval. Let us call this zero ξ, then

$$0 = g^{(n)}(\xi) = f^{(n)}(\xi) - p^{(n)}(\xi) = f^{(n)}(\xi) - f[x_0, \ldots, x_n]n!$$

This proves the theorem.

2.18: A note on notation

Historically divided differences are denoted by

$$f[x_0, \ldots, x_n]$$

The relation to differentiation operators motivates to write them in operator notation

$$[x_0, \ldots, x_n]f$$

The bracket denotes the n-th difference operator.

In this course we stick to the traditional notation.

2.19: Product rule analogon

Product rule for differentiation:

$$(f \cdot g)' = f' \cdot g + f \cdot g'$$

Product rule for difference operators:

$$(f \cdot g)[x_0, x_1] = f[x_0, x_1] \cdot g[x_1] + f[x_0] \cdot g[x_0, x_1]$$

The higher order case gives a generalization of the Leibnitz rule for higher order derivatives:

$$(f \cdot g)[x_0, \ldots, x_n] = \sum_{j=0}^{n} f[x_0, \ldots, x_j] \cdot g[x_j, \ldots, x_n]$$

2.20: Hermite Interpolation

Hermite interpolation aims to construct a polynomial, which not only interpolates the function f at given points, but also (higher) derivatives at those points have to coincide.

Example.
Consider $f(x) = \mathrm{sinc}(x) = \frac{\sin x}{x}$*.*
We want to construct a polynomial $p \in \mathcal{P}^5$ *such that*

$$\begin{array}{ccc} f(0) = 1 & f'(0) = 0 & \\ f(\pi) = 0 & f'(\pi) = -\frac{1}{\pi} & f''(\pi) = \frac{2}{\pi^2} \\ f(2\pi) = 0 & & \end{array}$$

For this end we allow multiple knots $X = \{0, 0, \pi, \pi, \pi, 2\pi\}$*.*

2.21: Hermite Interpolation (Cont.)

... and set up the interpolation scheme. All entries of the type $f[x_i, x_i]$ are replaced by the derivative $f'(x_i)$ as

$$\lim_{x_1 \to x_0} f[x_0, x_1] = \lim_{x_1 \to x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0} = f'(x_0)$$

We leave the remaining computation of p as an exercise.

2.22: Error Theorem

We discuss now a couple of expressions for the interpolation error. p denotes here always the polynomial interpolating f at the knots $X = \{x_0, \ldots, x_n\}$.

Theorem.

$$f(x) - p(x) = f[x_0, \ldots, x_n, x] \prod_{k=0}^{n} (x - x_k)$$

For the proof consider first the case $x = x_k$. Both sides are 0 due to the interpolation property of p. Then we take $x \in \mathbb{R} \setminus X$. Define $\hat{X} := X \cup \{x\}$. Denote by p_{n+1} the interpolation polynomial on $\hat{X}$.
$p_{n+1} = p(x) + f[x_0, \ldots, x_n, x] \prod_{k=0}^{n}(x - x_k)$.

Consequently, $f(x) - p(x) = f(x) - \big(p_{n+1}(x) - f[x_0, \ldots, x_n, x] \prod_{k=0}^{n}(x - x_k)\big)$.

As $f(x) = p_{n+1}(x)$ (for this particular x, the theorem is proven.

2.23: Error Theorem (Cont.)

Combining this theorem with the relation between finite differences and higher derivatives gives

Theorem. *There exists a* $\xi \in [x_{\min}, x_{\max}]$ *such that*

$$f(x) - p(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} \underbrace{\prod_{k=0}^{n} (x - x_k)}_{=:\omega_{n+1}(x)}$$

with $x_{\min} = \min\{x_0, \ldots, x_n, x\}$ *and* $x_{\max}$ *correspondingly.*

2.24: Error Estimate

This gives the following estimates

$$\|f-p\|_\infty \le \frac{\|f^{(n+1)}\|_\infty}{(n+1)!}\|\omega_{n+1}\|_\infty$$

with the $\max$-norm over $[a,b]$.

2.25: Best interpolation

There is a freedom when choosing interpolation points $x_i \in [a, b]$.

Goal: Find among all sets $X = \{x_0, \dots, x_n\} \subset [a, b]$ a set X^* such that

$$\|\omega_{X^*}\|_\infty \leq \|\omega_X\|_\infty$$

with

$$\omega_X = \prod_{k=0}^{n} (x - x_k)$$

To construct these optimal points we will use Tschebychev polynomials.