2.26 Tschebychev polynomials

We define functions

\[ T_n(x) = \cos(n \arccos(x)) \quad n \in \mathbb{N}_0 \]

and show in the next theorem, that these are polynomials, i.e. \( T_n \in \mathcal{P}^n \):

**Theorem.** With \( T_0(x) \equiv 1 \) and \( T_1(x) = x \) the functions \( T_n \) can be obtained from the 3-term recursion

\[ T_{n+1} = 2xT_n(x) - T_{n-1}(x) \]
2.27 Proof of Recursion Formula

The proof is by induction. The induction base is obvious. For the induction step, set \( \phi = \arccos x \) and apply the addition theorem:

\[
\cos((n \pm 1)\phi) = \cos(n\phi) \cos(\phi) \mp \sin(n\phi) \sin(\phi)
\]

Summing up these two terms gives

\[
T_{n+1}(x) + T_{n-1}(x) = 2xT_n(x)
\]

which completes the proof.

Note, \( T_n \) is an algebraic polynomial, the so-called Tschebychev polynomial. Its leading coefficient is \( 2^{n-1} \), thus

\[
T_n(x) = 2^{n-1}x^n + q(x) \quad \text{with} \quad q \in \mathcal{P}^{n-1}
\]
2.28 Tschebychev Points and Tschebychev Abscissae

The zeros of $T_n$ are called Tschebychev points $x^*_k$, with

$$x^*_k = \cos \left( \frac{2k + 1}{2n} \pi \right) \in [-1, 1] \quad 0 \leq k \leq n - 1$$

The extrem values $|T(\bar{x}_k)| = 1$ are attained at the so-called Tschebychev abscissae

$$\bar{x}_k = \cos \left( \frac{k}{n} \pi \right) \in [-1, 1] \quad 0 \leq k \leq n$$
2.29 The first three Tschebychev polynomials

\[ T_1(x) \]
\[ T_2(x) \]
\[ T_3(x) \]
2.30 Scaled Tschebychev polynomial

Let \( X^* = \{ x_k^* \mid 0 \leq k \leq n \} \) denote the set of the zeros of \( T_{n+1} \).

The polynomial
\[
\omega_{X^*}(x) := (x - x_0^*) \cdots (x - x_n^*)
\]
has leading coefficient one and the same zeros as \( T_{n+1} \).

Thus,
\[
\omega_{X^*}(x) = 2^{-n} T_{n+1}(x)
\]
2.31 Minimal Property of $T_n$

Theorem.

1. Let $P[-1, 1]$ have a leading coefficient $a_n \neq 0$, then there exists $\xi \in [-1, 1]$ such that

$$|P(\xi)| \geq \frac{|a_n|}{2^{n-1}}$$

2. Let $P_n^*[-1, 1]$ denote the set of polynomials of degree $n$ and leading coefficient 1. Then,

$$\|2^{-(n-1)}T_n\|_\infty \leq \min_{\omega \in P_n^*[-1, 1]} \|\omega\|_\infty$$
The proof of statement 1 is by contradiction: We assume the existence of a polynomial $p(x)$ with leading coefficient $a_n = 2^{-n}$ such that

$$|p(x)| < 1 \quad \forall x \in [-1, 1]$$

We then consider $p - T_n$ which is a polynomial of degree $n - 1$ as both polynomials have the same leading coefficient. For the Tschebychev abscissae $\bar{x}_k = \cos \left( \frac{k \pi}{n} \right)$ we observe

$$T_n(\bar{x}_{2k}) = 1 \land p(\bar{x}_{2k}) < 1 \quad \Rightarrow \quad p(\bar{x}_{2k}) - T_n(\bar{x}_{2k}) < 0$$

$$T_n(\bar{x}_{2k+1}) = -1 \land p(\bar{x}_{2k+1}) > -1 \quad \Rightarrow \quad p(\bar{x}_{2k}) - T_n(\bar{x}_{2k}) > 0$$

Consequently changes the difference polynomial its sign $n$-times in $[-1, 1]$. It must have $n$ roots, which would imply that it is the zero polynomial. But by construction $p - T_n$ cannot be zero.

The second statement is then a direct consequence.
2.33 Minimal Property of $T_n$ (cont.)

From this theorem we conclude, that an optimal placement of interpolation points for interpolation tasks on $[-1, 1]$ is to choose the Tschebychev points

$X^* = \{x^*_k|0 \leq k \leq n\}$.

For a general interval $[a, b]$ we apply a transformation

$[a, b] \rightarrow [-1, 1] \quad t \mapsto \tau = 2 \frac{t - a}{b - a} - 1$

See also Homework 2 for an example.