2.26 Tschebychev polynomials

We define functions

$$T_n(x) = \cos(n \arccos(x)) \quad n \in \mathbb{N}_0$$

and show in the next theorem, that these are polynomials, i.e. $T_n \in \mathcal{P}^n$:

Theorem. *With* $T_0(x) \equiv 1$ *and* $T_1(x) = x$ *the functions* T_n *can be obtained from the 3-term recursion*

$$T_{n+1} = 2xT_n(x) - T_{n-1}(x)$$

.

2.27 Proof of Recursion Formula

The proof is by induction. The induction base is obvious. For the induction step, set $\phi = \arccos x$ and apply the addition theorem:

$$\cos((n \pm 1)\phi) = \cos(n\phi)\cos(\phi) \mp \sin(n\phi)\sin(\phi)$$

Summing up these two terms gives

$$T_{n+1}(x) + T_{n-1}(x) = 2xT_n(x)$$

which completes the proof.

Note, T_n is an algebraic polynomial, the so-called Tschebychev polynomial. Its leading coefficient is 2^{n-1}, thus

$$T_n(x) = 2^{n-1}x^n + q(x) \text{ with } q \in \mathcal{P}^{n-1}$$

2.28 Tschebychev Points and Tschebychev Abscissae

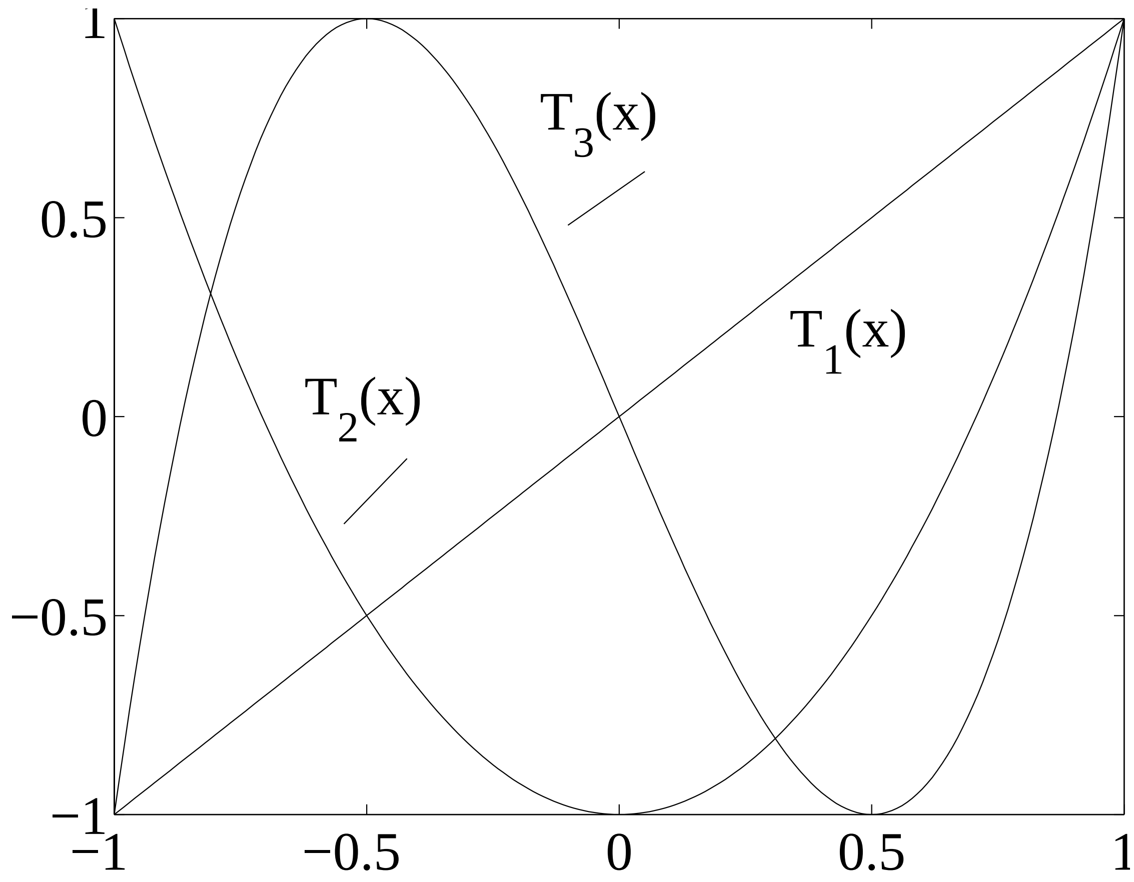
The zeros of T_n are called Tschebychev points x_k^*, with

$$x_k^* = \cos\left(\frac{2k+1}{2n}\pi\right) \in [-1,1] \qquad 0 \le k \le n-1$$

The extrem values $|T_(\bar{x}_k)| = 1$ are attained at the so-called Tschebychev abscissae

$$\bar{x}_k = \cos\left(\frac{k}{n}\pi\right) \in [-1,1] \qquad 0 \le k \le n$$

2.29 The first three Tschebychev polynomials



2.30 Scaled Tschebychev polynomial

Let $X^* = \{x_k^* | 0 \le k \le n\}$ denote the set of the zeros of T_{n+1}.

The polynomial

$$\omega_{X^*}(x) := (x - x_0^*) \cdots (x - x_n^*)$$

has leading coefficient one and the same zeros as T_{n+1}.

Thus,

$$\omega_{X^*}(x) = 2^{-n} T_{n+1}(x)$$

2.31 Minimal Property of T_n

Theorem.

1. *Let* $\mathcal{P}[-1,1]$ *have a leading coefficient* $a_n \neq 0$*, then there exists* $\xi \in [-1,1]$ *such that*
$$|P(\xi)| \geq \frac{|a_n|}{2^{n-1}}$$

2. *Let* $\mathcal{P}_n^*[-1,1]$ *denote the set of polynomials of degree* n *and leading coefficient* 1*. Then,*
$$\|2^{-(n-1)}T_n\|_\infty \leq \min_{\omega \in \mathcal{P}_n^*[-1,1]} \|\omega\|_\infty$$

2.32 Minimal Property of T_n (cont.)

The proof of statement 1 is by contradiction: We assume the existance of a polynomial $p(x)$ with leading coefficient $a_n = 2^{-n}$ such that

$$|p(x)| < 1 \quad \forall x \in [-1, 1]$$

We then consider $p - T_n$ which is a polynomial of degree $n-1$ as both polynomials have the same leading coefficient. For the Tschebychev abscissae $\bar{x}_k = \cos\left(\frac{k}{n}\pi\right)$ we observe

$$T_n(\bar{x}_{2k}) = 1 \wedge p(\bar{x}_{2k}) < 1 \quad \Rightarrow \quad p(\bar{x}_{2k}) - T_n(\bar{x}_{2k}) < 0$$

$$T_n(\bar{x}_{2k+1}) = -1 \wedge p(\bar{x}_{2k+1}) > -1 \quad \Rightarrow \quad p(\bar{x}_{2k}) - T_n(\bar{x}_{2k}) > 0$$

Consequently changes the difference polynomial its sign n-times in $[-1, 1]$. It must have n roots, which would imply that it is the zero polynomial. But by construction $p - T_n$ cannot be zero.

The second statement is then a direct consequence.

2.33 Minimal Property of T_n (cont.)

From this theorem we conclude, that an optimal placement of interpolation points for interpolation tasks on $[-1, 1]$ is to choose the Tschebychev points $X^* = \{x_k^* | 0 \leq k \leq n\}$.

For a general interval $[a, b]$ we apply a transformation

$$[a, b] \to [-1, 1] \qquad t \mapsto \tau = 2\,\frac{t - a}{b - a} - 1$$

See also Homework 2 for an example.