3. Best Approximations: Existence, Uniqueness, Characterization

The setting in this unit:

$$\left.\begin{array}{c}\mathcal{F} \text{ normed linear space} \\ \{\} \neq \mathcal{S} \subset \mathcal{F}\end{array}\right\} \text{Find for a given } f \in \mathcal{F}\backslash\mathcal{S} \texttt{ a } s^* \in \mathcal{S}$$

such that

$$\|s^* - f\| = \inf_{s \in \mathcal{S}} \|s - f\|$$

The minimal distance between f and the set $\mathcal{S}$ is denoted by

$$\eta(f, \mathcal{S}) := \inf_{s \in \mathcal{S}} \|s - f\|$$

We set up conditions for the existence and uniqueness of a best approximation s^*.

3.1 Norm is continuous

Definition. *A functional is* $\varphi : \mathcal{F} \to \mathcal{R}$ *is called continuous in* $u \in \mathcal{F}$ *if for all convergent sequences* $\{u_n\}$ *with* $\lim_{n\to\infty} u_n = u$ *the property*

$$\lim_{n\to\infty} \varphi(u_n) = \varphi(u)$$

holds.

Theorem. *A norm is a continuous functional.*

Proof: We consider the normed space $(\mathcal{F}, \|\cdot\|)$ and a sequence $\{u_n\}$ with $\lim_{n\to\infty} u_n = u$, i.e. $\|u_n - u\| \to 0$. From the triangular inequality for norms we conclude: $|\|u_n\| - \|u\|| \leq \|u_n - u\| \to 0$ ■

3.2 Existence for compact sets

Theorem.
Let $\mathcal{S} \subset \mathcal{F}$ be compact, then there exists for every $f \in \mathcal{F}$ a best approximation $s^* \in \mathcal{S}$.

Proof: $f \in \mathcal{F}, \varphi_f : \mathcal{F} \to [0, \infty]$ with $\varphi_f(v) := \|v - f\|$.

φ_f is continuous in $\mathcal{F}$. Consequently, it attains a maximum and a minimum on a compact set, i.e. $\exists s^*$ with

$$\varphi_f(s^*) = \|s^* - f\| \leq \|s - f\| = \varphi_f(s) \quad \forall s \in \mathcal{S}$$

3.3 Existence for closed sets in finite dimensional spaces

Theorem.
Let $\mathcal{F}$ be finite dimensional and $\mathcal{S} \subset \mathcal{F}$ be closed, then there exists for every $f \in \mathcal{F}$ a best approximation $s^* \in \mathcal{S}$.

Proof: Take any $s_0 \in \mathcal{S}$ and define for a given $f \in \mathcal{F}$ the set

$$\mathcal{S}_0 := \mathcal{S} \cup \{v \in \mathcal{F} \,, : \; \|v - f\| \leq \|s_0 - f\|\} \subset \mathcal{F}$$

It is closed and bounded, thus compact (as $\mathcal{F}$ is finite dimensional).

By the theorem before, there exists an s_0^* which is a best approximation to f in $\mathcal{S}_0 \subset \mathcal{S}$.

We check now if there is in $\mathcal{S} \backslash \mathcal{S}_0$ an element nearer to f:

Let $s \in \mathcal{S} \backslash \mathcal{S}_0$:

$$\|s - f\| < \|s_0 - f\| \geq \|s_0^* - f\|$$

This implies $s^* = s_0^* \in \mathcal{S}_0 \subset \mathcal{S}$ is a best approximation. ■

3.4 Existence for closed sets (Cont.)

The last theorem can be generalized to a situation very common in practice

Theorem.
Let $\mathcal{F}$ be finite dimensional and $\mathcal{R} \subset \mathcal{F}$ a finite dimensional subspace. Then, if $\mathcal{S} \subset \mathcal{R}$ is closed exists for every $f \in \mathcal{F}$ a best approximation $s^* \in \mathcal{S}$.

Proof: Let $\mathcal{R} = \mathrm{span}\{r_1, \ldots, r_n\}$ and define $\mathcal{R}_f = \mathrm{span}\{f, r_1, \ldots, r_n\} \subset \mathcal{F}$. The previous theorem then, completes the proof. ∎

Corollary.
Let $\mathcal{S} \subset \mathcal{F}$ be a finite dimensional subspace of $\mathcal{F}$. Then, every $f \in \mathcal{F}$ has a bestapproximation $s^* \in \mathcal{S}$.

As an example take $\mathcal{F} = \mathcal{C}[a, b]$ and $\mathcal{S} = \mathcal{P}^n$, the space of all polynomials of max degree n.

3.5 Euclidean Spaces and Parallelogram Identity

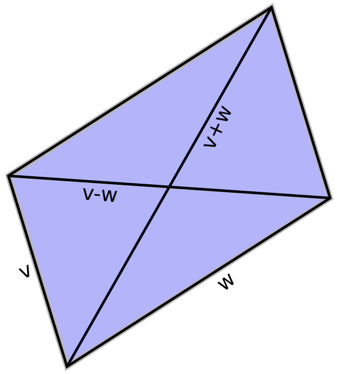
Eucledian spaces have an inner (scalar) product $(\cdot, \cdot)$ which induces a norm:

$$\|u\| := (u, u)^{\frac{1}{2}}$$

On the other hand, given a norm, can we find out, if there is an inner product which induces this norm?

The answer is the validity of the parallelogram equation

$$\|v + w\|^2 + \|v - w\|^2 = \|v\|^2 + \|w\|^2$$



This is shown in the next theorem.

3.6 Euclidean Spaces and Parallelogram Identity

Theorem. *Jordan – von Neumann*
Let $\mathcal{F}$ be a linear space with a norm having the property $\|v+w\|^2 + \|v-w\|^2 = 2\|v\|^2 + 2\|w\|^2 \quad \forall v, w \in \mathcal{F}$, then there exists an inner product $(\cdot,\cdot)$ in $\mathcal{F}$ such that $\|u\| := (u,u)^{\frac{1}{2}}$.

We define $(v,w) := \frac{1}{4}\left(\|v+w\|^2 - \|v-w\|^2\right)$ and show that this functions saftisfies the condition for an inner product:
Positivity: $(v,v) = \frac{1}{4}\|v\|^2 \geq 0$
Symmetry: obvious
Linearity: First we conclude from the definition $(-v,w) = -(v,w)$ and also $(0,w) = 0$. Then, by applying the parallelogram identity twice:

$$
\begin{aligned}
(u,w) + (v,w) &= \frac{1}{4}\left(\|u+w\|^2 + \|v+w\|^2 - (\|u-w\|^2 + \|v+w\|^2)\right) \\
&= frac14\left(\frac{1}{2}(\|u+w+v+w\|^2 + \|u+w-v-w\|^2 - \|u-w+v-w\|^2 - \|u-w-v+w\|^2)\right)
\end{aligned}
$$

3.7 Euclidean Spaces and Parallelogram Identity (Cont.)

$$
\begin{aligned}
(u,w)+(v,w) &= \frac{1}{4}\left(\frac{1}{2}(\|u+w+v+w\|^2+\|u+w-v-w\|^2-\|u-w+v-w\|^2-\|u-w-v+w\|^2)\right)\\
&= \frac{1}{4}\left(2\|\frac{u+v}{2}+w\|^2-2\|\frac{u+v}{2}-w\|^2\right)\\
&= 2(\frac{u+v}{2},w)
\end{aligned}
$$

As this holds for all $v\in\mathcal{F}$ so also for $v=0$, from which we conclude scalability by two: $2(\frac{u}{2},w)=(u,w)$. Using this and the derivation above results in $(u+v,w)=(u,w)+(v,w)$ and from $v=u$ and induction $(mu,w)=m(u,w)\quad\forall m\in\mathbb{Z}$. From the scalability by two and induction we also obtain $\frac{1}{2^n}(u,w)=(\frac{1}{2^n}u,w)\quad\forall n\in\mathbb{N}$. This can be combined to

$$
(\alpha u,w)=\alpha(u,w)\quad\forall\alpha=m+\sum_{k=1}^{n}\frac{\alpha_k}{2^k}\quad m\in\mathbb{Z},n\in\mathbb{N},\alpha_k\in 0,1.
$$

Every element $\alpha\in\mathbb{Q}$ can be expressed in this form (cf. binary numbers) . As $\mathbb{Q}$ is dense in $\mathbb{R}$, linearity $(\alpha u+\beta v,w)=\alpha(u,w)+\beta(v,w)$ follows. ■