3.7 Euclidean Spaces and Parallelogram Identity (Cont.)

\[(u, w) + (v, w) = \frac{1}{4} \left( \frac{1}{2}(\|u + w + v + w\|^2 + \|u + w - v - w\|^2 - \|u - w + v - w\|^2 - \|u - w - v + w\|^2) \right) \]

\[= \frac{1}{4} \left( 2\|\frac{u + v}{2} + w\|^2 - 2\|\frac{u + v}{2} - w\|^2 \right) \]

\[= 2\left(\frac{u + v}{2}, w\right) \]

As this holds for all \(v \in \mathcal{F}\) so also for \(v = 0\), from which we conclude scalability by two: \(2\left(\frac{u}{2}, w\right) = (u, w)\). Using this and the derivation above results in \((u + v, w) = (u, w) + (v, w)\) and from \(v = u\) and induction \((mu, w) = m(u, w) \quad \forall m \in \mathbb{Z}\).

From the scalability by two and induction we also obtain \(2^{-n}(u, w) = (2^{-n}u, w) \quad \forall n \in \mathbb{N}\). This can be combined to

\[(\alpha u, w) = \alpha(u, w) \quad \forall \alpha = m + \sum_{k=1}^{n} \frac{\alpha_k}{2^k} \quad m \in \mathbb{Z}, n \in \mathbb{N}, \alpha_k \in 0, 1.\]

Every element \(\alpha \in \mathbb{Q}\) can be expressed in this form (cf. binary numbers). As \(\mathbb{Q}\) is dense in \(\mathbb{R}\), linearity

\[(\alpha u + \beta v, w) = \alpha(u, w) + \beta(v, w)\] follows.

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3.8 Inner products and Continuity

**Theorem.**

*Every inner product is continuous (w.r.t. its induced norm).*

Proof: Consider two convergent sequences in $\mathcal{F}$: $v_n \to v$ and $w_n \to w$. Then,

\[
(v_n, w_n) = \frac{1}{4} \left( \|v_n + w_n\|^2 - \|v_n - w_n\|^2 \right) \to \frac{1}{4} \left( \|v + w\|^2 - \|v - w\|^2 \right) = (v, w)
\]

\[
\square
\]

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3.9 Convex sets

In the following convex sets play an important role

**Definition.**

A non empty set $\mathcal{K} \subset \mathcal{F}$ is called convex if for all $u, v \in \mathcal{K}$

$$\{\lambda u + (1 - \lambda)v : \lambda \in [0, 1]\} \subset \mathcal{K}.$$

It is called strictly convex if

$$\{\lambda u + (1 - \lambda)v : \lambda \in (0, 1)\} \subset \mathcal{K} \setminus \delta \mathcal{K},$$

where $\delta \mathcal{K}$ denotes the boundary of $\mathcal{K}$.
3.10 Best approximations in Hilbert spaces

Recall: A complete Euclidean space is called a Hilbert space.

**Theorem.** Let $\mathcal{F}$ be a Hilbert space with a norm $\|u\| := (u,u)^{\frac{1}{2}}$ and $S \subset \mathcal{F}$ be a closed and convex set. Then there exists for every $f \in \mathcal{F}$ a best approximation $s^* \in S$.

: Proof: Let $(s_n)_{n \in \mathbb{N}} \subset S$ be a minimal sequence, i.e. $\|s_n - f\| \to \eta(f, S) = \inf_{s \in S} \|s - f\|$ as $n \to \infty$. By the parallelogram identity we get

$$
\|s_n - s_m\|^2 = \|s_n - f -(s_m - f)\|^2 = 2\|s_n - f\|^2 + 2\|s_m - f\|^2 - 4\frac{s_n + s_m}{2} - f\|^2 \\
\leq 2\|s_n - f\|^2 + 2\|s_m - f\|^2 - 4\eta(f, S)
$$

(Here we used convexity of $S$.) As the last expression converges to zero, $\|s_n - s_m\| \to 0$. Consequently, $(s_n)_{n \in \mathbb{N}}$ is a Cauchy sequence. As $\mathcal{F}$ is complete, it converges to a point $s^*$, which is in $S$ as the set was assumed to be closed. $\blacksquare$

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3.11 The set of best approximations

**Theorem.** Let $S \subset \mathcal{F}$ be convex and $f \in \mathcal{F}$. Then the set of best approximations

$$S^* := S^*(f, S) := \{s^* \in S ; \|s^* - f\| = \eta(f, S)\}$$

is convex.

The non convex $S$ situation is demonstrated here:
3.12 The set of best approximations (Cont.)

Proof: Let $s_1^*$ and $s_2^*$ be two best approximations to $f$ and define $s^*_\lambda := \lambda s_1^* + (1 - \lambda) s_2^* \in S$. We have to show that also $s^*_\lambda$ is a best approximation.

\[
\|s^*_\lambda - f\| = \|\lambda s_1^* + (1 - \lambda) s_2^* - (\lambda + (1 - \lambda)) f\|
\[
= \|\lambda(s_1^* - f) + (1 - \lambda)(s_2^* - f)\|
\[
\leq \lambda \|s_1^* - f\| + (1 - \lambda) \|s_2^* - f\| = \eta(f, S)
\]

Example.

$S = \{\|x\|_\infty = 1\} \subset \mathcal{F} = \mathbb{R}^2$

and $f = (2, 0)$. 

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