3.8 Inner products and Continuity

**Theorem.**

*Every inner product is continuous (w.r.t. its induced norm).*

Proof: Consider two convergent sequences in $\mathcal{F}$: $v_n \to v$ and $w_n \to w$. Then,

$$(v_n, w_n) = \frac{1}{4} \left( \|v_n + w_n\|^2 - \|v_n - w_n\|^2 \right) \to \frac{1}{4} \left( \|v + w\|^2 - \|v - w\|^2 \right) = (v, w)$$

$\blacksquare$
3.9 Convex sets

In the following convex sets play an important role

**Definition.**

A non empty set $\mathcal{K} \subset \mathcal{F}$ is called convex if for all $u, v \in \mathcal{K}$

$$\{\lambda u + (1 - \lambda)v : \lambda \in [0, 1]\} \subset \mathcal{K}.$$

*It is called strictly convex if*

$$\{\lambda u + (1 - \lambda)v : \lambda \in (0, 1)\} \subset \mathcal{K} \setminus \delta \mathcal{K},$$

*where $\delta \mathcal{K}$ denotes the boundary of $\mathcal{K}$.***
3.10 Best approximations in Hilbert spaces

Recall: A complete Euclidean space is called a Hilbert space.

**Theorem.** Let $\mathcal{F}$ be a Hilbert space with a norm $\|u\| := (u, u)^{1/2}$ and $S \subset \mathcal{F}$ be a closed and convex set. Then there exists for every $f \in \mathcal{F}$ a best approximation $s^* \in S$.

**Proof:** Let $(s_n)_{n \in \mathbb{N}} \subset S$ be a minimal sequence, i.e. $\|s_n - f\| \to \eta(f, S) = \inf_{s \in S} \|s - f\|$ as $n \to \infty$. By the parallelogram identity we get

$$\|s_n - s_m\|^2 = \|s_n - f - (s_m - f)\|^2 = 2\|s_n - f\|^2 + 2\|s_m - f\|^2 - 4\frac{s_n + s_m}{2} - f\|^2$$

$$\leq 2\|s_n - f\|^2 + 2\|s_m - f\|^2 - 4\eta(f, S)$$

(Here we used convexity of $S$.) As the last expression converges to zero, $\|s_n - s_m\| \to 0$. Consequently, $(s_n)_{n \in \mathbb{N}}$ is a Cauchy sequence. As $\mathcal{F}$ is complete, it converges to a point $s^*$, which is in $S$ as the set was assumed to be closed. □

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3.11 The set of best approximations

**Theorem.** Let $S \subset \mathcal{F}$ be convex and $f \in \mathcal{F}$. Then the set of best approximations

$$S^* := S^*(f, S) := \{ s^* \in S ; \| s^* - f \| = \eta(f, S) \}$$

is convex.

The non convex $S$ situation is demonstrated here:

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3.12 The set of best approximations (Cont.)

Proof: Let \( s_1^* \) and \( s_2^* \) be two best approximations to \( f \) and define \( s_\lambda^* := \lambda s_1^* + (1 - \lambda) s_2^* \in S \). We have to show that also \( s_\lambda^* \) is a best approximation.

\[
\|s_\lambda^* - f\| = \|\lambda s_1^* + (1 - \lambda) s_2^* - (\lambda + (1 - \lambda)) f\|
\]
\[
= \|\lambda (s_1^* - f) + (1 - \lambda) (s_2^* - f)\|
\]
\[
\leq \lambda \|s_1^* - f\| + (1 - \lambda) \|s_2^* - f\| = \eta(f, S)
\]

Example.

\( S = \{\|x\|_\infty = 1\} \subset \mathcal{F} = \mathbb{R}^2 \)

and \( f = (2, 0) \).
3.13 Convex functional, strictly convex functional, strictly convex norm

**Definition.** Let $\lambda \in (0, 1)$:

- A functional $\varphi : F \to \mathbb{R}$ is called convex if

  $$
  \varphi(\lambda u + (1 - \lambda)v) \leq \lambda \varphi(u) + (1 - \lambda)\varphi(v)
  $$

  If "$\leq$" can be replaced by "$<$" it is called strictly convex.

- A norm $\| \cdot \|$ is called strictly convex if the unit ball $\{u \in F : \|u\| \leq 1\}$ is a strictly convex set.

Note, a norm is always a convex functional (due to triangular inequality) $\| \lambda u + (1 - \lambda)v \| \leq \lambda \|u\| + (1 - \lambda)\|v\|$ but by setting $u = \alpha v$ we easily see that it can not be a strictly convex functional. This explains, why we had to define this term differently for norms.

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3.14 Characterization of strictly convex norms

**Theorem.**

*The following statements are equivalent*

(a) $\| \cdot \|$ *is strictly convex.*

(b) *For all* $u \neq v$ *with* $\|u\| = \|v\| = 1$: $\|u + v\| < 2$

(c) $\|u + v\| = \|u\| + \|v\|$ $\Rightarrow u = \alpha v$ $\alpha \in \mathbb{R}$
3.15 Uniqueness of best approximations - strict convex norm

**Theorem.**
Let $(\mathcal{F}, \|\cdot\|)$ be a normed space with a strictly convex norm and let $S$ be a convex subset. Then there exists for every $f \in \mathcal{F}$ a unique best approximation $s^* \in S$.

**Proof:** Let $s_1^*, s_2^*$ be two best approximations and assume $s_1^* \neq s_2^*$. Then, $\|s_1^* - f\| = \|s_2^* - f\| = \eta(f, S)$. Consider $s^* = \frac{s_1^* + s_2^*}{2} \in S$ (convexity). Then by the strict convexity of the norm ($\lambda = \frac{1}{2}$) we conclude

$$\|s^* - f\| = \left\| \frac{s_1^* + s_2^*}{2} - f \right\| = \frac{1}{2} \left\| s_1^* - f + s_2^* - f \right\| < \|s_1^* - f\|$$

which contradicts the assumption that $s_1^*$ is a best approximation. ■
3.16: Examples of normed spaces with strictly convex norms

**Theorem.**

*The Euclidean norm* $\| \cdot \| = (\cdot, \cdot)^{\frac{1}{2}}$ is strictly convex.

Proof: Parallelogram equation is valid (Jordan-von Neuman Theorem) $\| u + v \|^2 + \| u - v \|^2 = 2\| u \|^2 + 2\| v \|^2$ and consequently also $\left\| \frac{u+v}{2} \right\|^2 + \left\| \frac{u-v}{2} \right\|^2 = \frac{\| u \|^2}{2} + \frac{\| v \|^2}{2}$. Now, take $u \neq v$ with $\| u \| = \| v \| = 1$. Then, $\left\| \frac{u+v}{2} \right\|^2 < \| u \|^2$.

If we then consider an $u$ with $\| u \| = 1$ we obtain $\| u + v \| < 2$, which is equivalent to strict convexity by one of the previous theorems.

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3.17: More examples of normed spaces with/without strictly convex norms

see blackboard