3.8 Inner products and Continuity

Theorem.
Every inner product is continuous (w.r.t. its induced norm).

Proof: Consider two convergent sequences in $\mathcal{F}$: $v_n \to v$ and $w_n \to w$.
Then,

$$(v_n, w_n) = \frac{1}{4}\left(\|v_n + w_n\|^2 - \|v_n - w_n\|^2\right) \to \frac{1}{4}\left(\|v + w\|^2 - \|v - w\|^2\right) = (v, w)$$

■

3.9 Convex sets

In the following convex sets play an important role

Definition.
A non empty set $\mathcal{K} \subset \mathcal{F}$ is called convex if for all $u, v \in \mathcal{K}$

$$\{\lambda u + (1-\lambda)v \ : \ \lambda \in [0,1]\} \subset \mathcal{K}.$$

It is called strictly convex if

$$\{\lambda u + (1-\lambda)v \ : \ \lambda \in (0,1)\} \subset \mathcal{K} \backslash \delta\mathcal{K},$$

where $\delta\mathcal{K}$ denotes the boundary of $\mathcal{K}$.

3.10 Best approximations in Hilbert spaces

Recall: A complete Euclidean space is called a Hilbert space.

Theorem. *Let $\mathcal{F}$ be a Hilbert space with a norm $\|u\| := (u,u)^{\frac{1}{2}}$ and $S \subset \mathcal{F}$ be a closed and convex set. Then there exists for every $f \in \mathcal{F}$ a best approximation $s^* \in S$.*

: Proof: Let $(s_n)_{n\in\mathbb{N}} \subset S$ be a minimal sequence, i.e. $\|s_n - f\| \to \eta(f,S) = \inf_{s\in S} \|s-f\|$ as $n \to \infty$.
By the parallelogram identity we get

$$\begin{aligned} \|s_n - s_m\|^2 &= \|s_n - f - (s_m - f)\|^2 = 2\|s_n - f\|^2 + 2\|s_m - f\|^2 - 4\|\frac{s_n + s_m}{2} - f\|^2 \\ &\leq 2\|s_n - f\|^2 + 2\|s_m - f\|^2 - 4\eta(f,S) \end{aligned}$$

(Here we used convexity of $\mathcal{S}$.) As the last expression converges to zero, $\|s_n - s_m\| \to 0$. Consequently, $(s_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. As $\mathcal{F}$ is complete, it converges to a point s^*, which is in $\mathcal{S}$ as the set was assumed to be closed. ∎

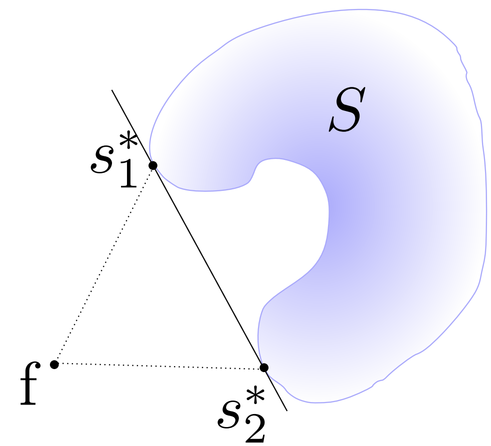
3.11 The set of best approximations

Theorem. *Let $\mathcal{S} \subset \mathcal{F}$ be convex and $f \in \mathcal{F}$. Then the set of bestapproximations*

$$S^* := S^*(f, S) := \{s^* \in \mathcal{S} \;\; ; \;\; \|s^* - f\| = \eta(f, \mathcal{S})\}$$

is convex.

The non convex $\mathcal{S}$ situation is demonstrated here:



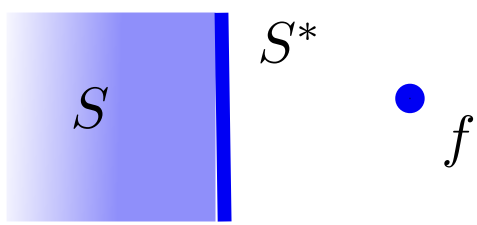
3.12 The set of best approximations (Cont.)

Proof: Let s_1^* and s_2^* be two best approximations to f and define $s_\lambda^* := \lambda s_1^* + (1-\lambda)s_2^* \in \mathcal{S}$. We have to show that also s_λ^* is a best approximation.

$$\begin{aligned}\|s_\lambda^* - f\| &= \|\lambda s_1^* + (1-\lambda)s_2^* - (\lambda + (1-\lambda))f\| \\ &= \|\lambda(s_1^* - f) + (1-\lambda)(s_2^* - f)\| \\ &\leq \lambda\|s_1^* - f\| + (1-\lambda)\|s_2^* - f\| = \eta(f, \mathcal{S})\end{aligned}$$

■

Example.
$\mathcal{S} = \{\|x\|_\infty = 1\} \subset \mathcal{F} = \mathbb{R}^2$
and $f = (2, 0)$.



3.13 Convex functional, strictly convex functional, strictly convex norm

Definition. *Let* $\lambda \in (0, 1)$*:*

- *A functional* $\varphi : \mathcal{F} \to \mathbb{R}$ *is called convex if*

$$\varphi(\lambda u + (1 - \lambda)v) \leq \lambda\varphi(u) + (1 - \lambda)\varphi(v)$$

  If "$\leq$*" can be replaced by "*$<$*" it is called strictly convex.*

- *A norm* $\|\cdot\|$ *is called strictly convex if the unit ball* $\{u \in \mathcal{F} : \|u\| \leq 1\}$ *is a strictly convex set.*

Note, a norm is always a convex functional (due to triangular inequality) $\|\lambda u + (1 - \lambda)v\| \leq \lambda\|u\| + (1 - \lambda)\|v\|$ but by setting $u = \alpha v$ we easily see that it can not be a strictly convex functional. This explains, why we had to define this term differently for norms.

3.14 Characterization of strictly convex norms

Theorem.
The following statements are equivalent

(a) $\|\cdot\|$ *is strictly convex.*

(b) For all $u \neq v$ *with* $\|u\| = \|v\| = 1$*:* $\qquad \|u + v\| < 2$

(c) $\|u + v\| = \|u\| + \|v\| \Rightarrow u = \alpha v \;\; \alpha \in \mathbb{R}$

3.15 Uniqueness of best approximations - strict convex norm

Theorem.
Let $(\mathcal{F}, \|.\|)$ be a normed space with a strictly convex norm and let $\mathcal{S}$ be a convex subset. Then there exists for every $f \in \mathcal{F}$ a unique bestapproximation $s^* \in S$.

Proof: Let s_1^*, s_2^* be two best approximations and assume $s_1^* \neq s_2^*$. Then, $\|s_1^* - f\| = \|s_2^* - f\| = \eta(f, S)$. Consider $s^* = \frac{s_1^*+s_2^*}{2} \in S$ (convexity). Then by the strict convexity of the norm ($\lambda = \frac{1}{2}$) we conclude
$\|s^* - f\| = \|\frac{s_1^*+s_2^*}{2} - f\| = \|\frac{s_1^*-f+s_2^*-f}{2}\| < \|s_1^* - f\|$ which contradicts the assumption that s_1^* is a best approximation. ■

3.16: Examples of normed spaces with strictly convex norms

Theorem.
The Euclidean norm $\|\cdot\| = (\cdot,\cdot)^{\frac{1}{2}}$ *is strictly convex.*

Proof: Parallelogram equation is valid (Jordan-von Neuman Theorem) $\|u+v\|^2 + \|u-v\|^2 = 2\|u\|^2 + 2\|v\|^2$ and consequently also $\|\frac{u+v}{2}\|^2 + \|\frac{u-v}{2}\|^2 = \frac{\|u\|^2}{2} + \frac{\|v\|^2}{2}$. Now, take $u \neq v$ with $\|u\| = \|v\| = 1$. Then, $\|\frac{u+v}{2}\|^2 < \|u\|^2$. If we then consider an u with $\|u\| = 1$ we obtain $\|u+v\| < 2$, which is equivalent to strict convexity by one of the previous theorems.

3.17: More examples of normed spaces with/without strictly convex norms

see blackboard