4: Direct Characterizations of Best Approximations

In this unit we characterize the (unique) best approximation. Note, that we cannot just differentiate the norm and find for a zero of the derivative as one does to find a minimum of a differentiable function. Most norms are not differentiable.

4.1: The Gâteaux derivative

Definition. *Let $\varphi : \mathcal{F} \to \mathbb{R}$ be a functional. Then*

$$\varphi'_+(u,v) := \lim_{h \to 0^+} \frac{\varphi(u+hv) - \varphi(u)}{h}$$

is called the Gâteaux derivative of φ at u in direction v.

Theorem. *If φ is convex then φ'_+ exists for all $u, v \in \mathcal{F}$ and*

$$-\varphi'_+(u,-v) \leq \varphi'_+(u,v)$$

– no proof – ■

4.2: Gâteaux derivative of convex functionals – Properties

Theorem.
Let $\varphi : \mathcal{F} \to \mathbb{R}$ be a convex functional. Then has its Gâteaux derivative the following properties for all $u, v, w \in \mathcal{F}$ and $\alpha \in \mathbb{R}$:

a) $\varphi'_+(u, \alpha v) = \alpha \varphi'_+(u, v)$ *(homegenity)*

b) $\varphi'_+(u, v + w) \leq \varphi'_+(u, v) + \varphi'_+(u, v)$ *(sublinearity)*

c) $\varphi'_+(u, \cdot)$ *is a convex function (in its second argument)*

Proof:

(a), (b) $\Rightarrow$ (a):

$$\varphi'_+(u, \lambda v + (1-\lambda) w) \overset{(b)}{\leq} \varphi'_+(u, \lambda v) + \varphi'_+(u, (1-\lambda) w) \overset{(a)}{=} \lambda \varphi'_+(u, v) + (1-\lambda) \varphi'_+(u, w)$$

4.3: Gâteaux derivative of convex functionals – Cont.

(b): $u + h(v+w) = \frac{1}{2}(u+2hv) + \frac{1}{2}(u+2hw)$

$$\begin{aligned}
\varphi'_+(u, v+w) &= \lim_{h\to 0^+} \frac{\varphi(u+h(v+w)) - \varphi(u)}{h} \leq \lim_{h\to 0^+} \frac{\frac{1}{2}\varphi(u+2hv) + \frac{1}{2}\varphi(u+2hw) - \varphi(u)}{h} \\
&\leq \lim_{h\to 0^+} \frac{1}{2h}(\varphi(u+2hv) - \varphi(u)) + \frac{1}{2h}(\varphi(u+2hw) - \varphi(u)) \\
&= \varphi'_+(u,v) + \varphi'_+(u,w)
\end{aligned}$$

(a): (easy) ■

We note also a chain rule for Gâteaux derivatives:

Theorem. $\varphi : \mathcal{F} \to \mathbb{R}, \quad F \in \mathcal{C}^1(\mathbb{R})$ *then:*

$$(F \circ \varphi)'_+(u,v) = F'(\varphi(u)) \cdot \varphi'_+(u,v)$$

$\circ$ *denotes the compostion of two functions:* $(f \circ g)(u) = f(g(u))$.

4.4: Minimum of convex functionals

Theorem. *Let $\varphi : \mathcal{F} \to \mathbb{R}$ be a convex funtional and $\mathcal{K} \subset \mathcal{F}$ be a convex set with $u_0 \in \mathcal{K}$. Then,*

$$\varphi(u_0) = \inf_{u \in \mathcal{K}} \varphi(u) \iff \varphi'_+(u_0, u - u_0) \geq 0 \ \ \forall u \in \mathcal{K}$$

Proof: - for the proof see your lecture notes – ∎

Note: $\|\cdot\|$ is a convex functional. Also the shifted norm $\varphi_f(v) := \|v - f\|$ is a convex functional. Furthermore it is continuous.

We denote the Gâteaux derivative of the norm by $\|\cdot\|'_+$.

4.6: Kolmogoroff Theorem - A characterization theorem

Theorem. *Let $f \in \mathcal{F}$ and let $\mathcal{S} \subset \mathcal{F}$ be a convex subset. Then the following statements are equivalent:*

a) $s^* \in \mathcal{S}$ is a best approximation to f

b) $\| \cdot \|'_+(s^* - f, s - s^*) \geq 0 \;\; \forall s \in \mathcal{S}$

Proof: $\varphi(u) := \|u - f\|$.

$$\begin{aligned}
\varphi'_+(s^*, s - s^*) &= \lim_{h \to 0+} \frac{1}{h}(\varphi(s^* + h(s - s^*)) - \varphi(s^*)) = \lim_{h \to 0+} \frac{1}{h}(\|s^* + h(s - s^*) - f\| - \|s^* - f\|) \\
&= \lim_{h \to 0+} \frac{1}{h}(\|(s^* - f) + h(s - s^*)\| - \|s^* - f\|) = \| \cdot \|'_+(s^* - f, s - s^*) \geq 0
\end{aligned}$$

The rest follows from the previous theorem (Minimum of convex functionals). ■