4: Direct Characterizations of Best Approximations

In this unit we characterize the (unique) best approximation. Note, that we cannot just differentiate the norm and find for a zero of the derivative as one does to find a minimum of a differentiable function. Most norms are not differentiable.
4.1: The Gâteaux derivative

**Definition.** Let \( \varphi : \mathcal{F} \to \mathbb{R} \) be a functional. Then

\[
\varphi'_+(u, v) := \lim_{h \to 0^+} \frac{\varphi(u + hv) - \varphi(u)}{h}
\]

is called the Gâteaux derivative of \( \varphi \) at \( u \) in direction \( v \).

**Theorem.** If \( \varphi \) is convex then \( \varphi'_+ \) exists for all \( u, v \in \mathcal{F} \) and

\[
-\varphi'_+(u, -v) \leq \varphi'_+(u, v)
\]

– no proof –
4.2: Gâteaux derivative of convex functionals – Properties

**Theorem.**

Let \( \varphi : \mathcal{F} \to \mathbb{R} \) be a convex functional. Then has its Gâteaux derivative the following properties for all \( u, v, w \in \mathcal{F} \) and \( \alpha \in \mathbb{R} \):

\[
a) \quad \varphi'_+(u, \alpha v) = \alpha \varphi'_+(u, v) \tag{homegenity} \]
\[
b) \quad \varphi'_+(u, v + w) \leq \varphi'_+(u, v) + \varphi'_+(u, v) \tag{sublinearity} \]
\[
c) \quad \varphi'_+(u, \cdot) \text{ is a convex function (in its second argument)}
\]

**Proof:**

(a), (b) \( \Rightarrow \) (a):

\[
\varphi'_+(u, \lambda v + (1 - \lambda)w) \leq \varphi'_+(u, \lambda v) + \varphi'_+(u, (1 - \lambda)w) \overset{(a)}{=} \lambda \varphi'_+(u, v) + (1 - \lambda)\varphi'_+(u, w)
\]

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4.3: Gâteaux derivative of convex functionals – Cont.

(b): \( u + h(v + w) = \frac{1}{2}(u + 2hv) + \frac{1}{2}(u + 2hw) \)

\[
\phi'(u, v + w) = \lim_{h \to 0^+} \frac{\phi(u + h(v + w)) - \phi(u)}{h} \leq \lim_{h \to 0^+} \frac{1}{2} \phi(u + 2hv) + \frac{1}{2} \phi(u + 2hw) - \phi(u)
\]

\[
\leq \lim_{h \to 0^+} \frac{1}{2h} (\phi(u + 2hv) - \phi(u)) + \frac{1}{2h} (\phi(u + 2hw) - \phi(u))
\]

\[= \phi'(u, v) + \phi'(u, w) \]

(a): (easy) \[\blacksquare\]

We note also a chain rule for Gâteaux derivatives:

**Theorem.** \( \phi : \mathcal{F} \to \mathbb{R}, \ F \in C^1(\mathbb{R}) \) then:

\[
(F \circ \phi)'_+(u, v) = F'(\phi(u)) \cdot \phi'_+(u, v)
\]

○ denotes the composition of two functions: \((f \circ g)(u) = f(g(u))\).

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4.4: Minimum of convex functionals

**Theorem.** Let $\varphi : \mathcal{F} \to \mathbb{R}$ be a convex functional and $\mathcal{K} \subset \mathcal{F}$ be a convex set with $u_0 \in \mathcal{K}$. Then,

$$\varphi(u_0) = \inf_{u \in \mathcal{K}} \varphi(u) \iff \varphi'(u_0, u - u_0) \geq 0 \quad \forall u \in \mathcal{K}$$

**Proof:** - for the proof see your lecture notes – ■

**Note:** $\| \cdot \|$ is a convex functional. Also the shifted norm $\varphi_f(v) := \|v - f\|$ is a convex functional. Furthermore it is continuous.

We denote the Gâteaux derivative of the norm by $\| \cdot \|'$.
4.6: Kolmogoroff Theorem - A characterization theorem

**Theorem.** Let \( f \in \mathcal{F} \) and let \( S \subset \mathcal{F} \) be a convex subset. Then the following statements are equivalent:

a) \( s^* \in S \) is a best approximation to \( f \)

b) \( \| \cdot \|'_{+}(s^* - f, s - s^*) \geq 0 \quad \forall s \in S \)

Proof: \( \varphi(u) := \|u - f\| \).

\[
\varphi'(s^*, s - s^*) = \lim_{h \to 0^+} \frac{1}{h} (\varphi(s^* + h(s - s^*)) - \varphi(s^*)) = \lim_{h \to 0^+} \frac{1}{h} (\|s^* + h(s - s^*) - f\| - \|s^* - f\|) \\
= \lim_{h \to 0^+} \frac{1}{h} (\|(s^* - f) + h(s - s^*)\| - \|s^* - f\|) = \| \cdot \|'_{+}(s^* - f, s - s^*) \geq 0
\]

The rest follows from the previous theorem (Minimum of convex functionals). □

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