Features of the Nyström Method for the Sherman-Lauricella Equation on Piecewise Smooth Contours

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Abstract

Let \( \Gamma \) be a simple closed contour in the complex plane \( \mathbb{C} \). The Sherman-Lauricella equation

\[
\omega(t) + \frac{1}{2\pi i} \int_{\Gamma} \omega(\tau) d\ln \left( \frac{\tau - t}{\tau - \bar{t}} \right) - \frac{1}{2\pi i} \int_{\Gamma} \omega(\tau) d\left( \frac{\tau - t}{\tau - \bar{t}} \right) = f(t), \quad t \in \Gamma.
\]

(1)

plays an important role in various applications. Thus it is used in radar imaging, in theory of viscous incompressible flows, as well as in plane elasticity and in other problems of mathematical physics [1, 2]. However, the solution of equation (1), if it exists, is not unique and there is no analytic formula to obtain it.

In this work we study the stability of an approximation method in the space \( L_2(\Gamma) \) for equation (1) in the case where contour \( \Gamma \) has corner points. More precisely, we consider the Nyström method based on composite Gauss-Legendre quadrature formula

\[
\int_0^1 u(s) ds \approx \sum_{l=0}^{n-1} \sum_{p=0}^{d-1} w_p u(s_{lp}) / n,
\]

(2)

where

\[
s_{lp} = \frac{l + \varepsilon_p}{n}, \quad l = 0, 1, \ldots, n - 1; \quad p = 0, 1, \ldots, d - 1,
\]

and \( 0 < \varepsilon_0 < \varepsilon_1, \ldots, < \varepsilon_d < 1 \) are the weights and the Gauss-Legendre points on the interval \([0,1]\).

It is worth noting that for smooth contours \( \Gamma \) the integral operators in the left-hand side of (1) are compact, and the method proposed is always stable. On the other hand, if \( \Gamma \) possesses angular points \( \tau_k, k = 1, 2, \ldots, m \), the stability of the method under consideration is connected to the invertibility of certain operators \( A_{\tau_k} \) from an algebra of Toeplitz operators. The operators
\( A_{\tau_k} \) depend on the above parameters \( \varepsilon_p \) and on the magnitude of the corresponding angles at the points \( \tau_k \). These operators \( A_{\tau_k} \) arise if one applies quadrature formulas similar to formulas (2)-(3) to Mellin operators with the kernels defined by the kernels of the integral operators in equation (1). They have a complicated structure and, at the moment, there is no analytic tools to study their invertibility. However, each such operator can be associated with the corresponding Nyström method for the Sherman-Laurichella equation on a special model curve \( \Gamma_0 \) with only one corner point. On the other hand, the stability of the method in those model situations is connected to the behaviour of condition numbers of systems of algebraic equations and it can be tested numerically. Thus, considering the sequences of the condition numbers we obtain certain information about the invertibility of the operators \( A_{\tau_k} \).

References
