Hermite interpolation is shown to be much more stable than Nyström interpolation in the context of solving classic Fredholm second kind integral equations of potential theory in two dimensions using panel-based Nyström discretization.


Key words: integral equations, interpolation, potential theory.

1 Nyström discretization and Nyström interpolation

A popular method for solving Fredholm second kind boundary integral equations

\[ \mu(t) + \int_{\Gamma} K(t, s) \mu(s) \, ds = r(t) \tag{1.1} \]

is Nyström discretization: The integral in (1.1) is discretized on the boundary $\Gamma$ according to a quadrature rule with nodes and weights $t_j$ and $w_j$, $j = 1, \ldots, N$, and the resulting semi-discrete equation for the unknown layer density $\mu(t)$ is enforced at the quadrature nodes. Upon solving the linear system

\[ \tilde{\mu}(t_j) + \sum_{k=1}^{N} K(t_j, t_k) \tilde{\mu}(t_k) w_k = r(t_j), \quad j = 1, \ldots, N, \tag{1.2} \]

one obtains a discretized approximation $\tilde{\mu}(t_j)$ to $\mu(t_j)$ whose convergence reflects that of the quadrature. For example, let the kernel function $K(t, s)$ and the right hand side $r(t)$ be smooth on a smooth $\Gamma$ and let $n$-point composite Gauss–Legendre quadrature with $m$ quadrature panels be used in (1.2) so that $N = mn$. Then the Euclidean relative error

\[ E(\tilde{\mu}, \mu, t) = \left( \frac{\sum_{j=1}^{N} |\tilde{\mu}(t_j) - \mu(t_j)|^2}{\sum_{j=1}^{N} |\mu(t_j)|^2} \right)^{1/2} \tag{1.3} \]

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decays as $1/N^{2n}$ when $m$ increases. The arguments $\tilde{\mu}$, $\mu$, and $t$ of $E(\cdot, \cdot, \cdot)$ denote vectors with $N$ entries $\tilde{\mu}(t_j)$, $\mu(t_j)$, and $t_j$.

Suppose now that, after having solved (1.2) for $\tilde{\mu}(t_j)$, we want approximations $\hat{\mu}(t)$ to $\mu(t)$ at arbitrary $t$, not just at $t_j$. Assume also that $r(t)$ can be accurately evaluated for arbitrary $t$. Then one can use Nyström interpolation

$$
\hat{\mu}(t) = r(t) - \sum_{k=1}^{N} K(t, t_k)\tilde{\mu}(t_k)w_k.
$$

Nyström interpolation has the theoretical advantage that it carries over the convergence properties of $\tilde{\mu}(t_j)$ to $\hat{\mu}(t)$. The entire solution process, involving discretization of (1.1), solution of the linear system (1.2), and interpolation in (1.4), is called the Nyström method [1].

Nyström interpolation is widely discussed and often also recommended, see [1, Chapter 4.1], [3, Chapter 4.2], [8, Chapter 10.5], [12, Chapter 18.1], and [2, 9, 11, 13]. Note that Nyström interpolation, in exact arithmetic, is accurate as soon as $K(t, s)\mu(s)$ is resolved by the basis functions in $s$ that underlie the quadrature. It is not necessary that $K(t, t), \mu(t)$, or $r(t)$ are resolved by any basis functions in $t$. An extreme example is when $K(t, s)$ is zero. Then $\tilde{\mu}(t)$ is equal to $r(t)$ and Nyström interpolation is accurate irrespective of the regularity of the right hand side.

Despite theoretically appealing properties, the performance of Nyström interpolation can be quite poor in practice. In particular this holds for many of the most common integral equations of potential theory where the kernel has a denominator that vanishes for $s = t$. The effect of cancellation as $t \to t_k$ in (1.4) can be arbitrarily large and completely destroy the accuracy of the interpolation. Strangely, this phenomenon does not seem to have been discussed in the literature. The present paper presents a competitive alternative to Nyström interpolation based on Hermite interpolation. The technique applies in a composite quadrature setting whenever $K(t, s)$ is smooth and has a smooth derivative with respect to $t$. For brevity, we only give two examples. One very simple for Laplace’s equation and one more involved for the biharmonic equation.

### 2 Two equations.

The standard double layer integral equation for Laplace’s equation with Dirichlet boundary conditions on a simply connected interior domain in two dimensions, see § 29 of [10], reads

$$
\mu(t) + \frac{1}{\pi} \int_{-\pi}^{\pi} 3 \left\{ \frac{z'(s)}{z(s) - z(t)} \right\} \mu(s) \, ds = 2f(t),
$$

where the boundary has positive orientation and is parameterized in the complex plane by $z(t)$, $t \in (-\pi, \pi]$, $z'(t) \equiv dz(t)/dt$, and $f(t)$ is the Dirichlet boundary data.

A Muskhelishvili complex variable integral equation for the two-dimensional interior stress problem, see Section 5 of [6], § 54 of [10], and also [4], can be
written

\[
\mu(t) = \frac{1}{2\pi i} \int_{-\pi}^{\pi} \left[ \frac{z'(s)\mu(s)}{z(s) - z(t)} - \frac{z'(t) z'(s)\mu(s)}{z'(t) (z(s) - z(t))} + \frac{z'(s)\mu(s)}{z(s) - z(t)} \right. \\
\left. \frac{-z''(t) (z(s) - z(t)) z'(s)\bar{\mu}(s)}{z'(t) (z(s) - z(t))^2} \right] ds + \frac{i}{A} \int_{-\pi}^{\pi} \Re \{ \tilde{z}(s) z'(s)\mu(s) \} \ ds = \\
-\frac{z''(t)}{z'(t)} \frac{1}{2\pi i} \int_{-\pi}^{\pi} \frac{(b(s) - b(t)) z'(s) \bar{\mu}\, ds}{\tilde{z}(s) - \tilde{z}(t)},
\]

where \( \tilde{z} \) is the complex conjugate of \( z \), \( A \) is the area of the elastic body, \( b(t) \) has the interpretation of prescribed (complex) traction along the boundary times the (complex) outward unit normal. The solution \( \mu(t) \), denoted \( \Phi(z(t)) \) in the solid mechanics literature \[10\], is the limit on the boundary of a function, analytic inside the body and simply related to the components of the stress tensor. In both (2.1) and (2.2) cancellation will inevitably occur as \( s \to t \).

We point out that the kernels of (2.1) and (2.2) have computable limits for \( s \to t \). These limits can be used in (1.2) for \( k = j \) and, apart from that, two points \( z(t_k) \) and \( z(t_j) \) will never be extremely close to each other. So cancellation is not a problem when (2.1) or (2.2) are used in (1.2). But when a discrete solution to (2.1) or (2.2) is used in (1.4), the point \( z(t) \) could be arbitrarily close to a point \( z(t_k) \) and cancellation is indeed a problem.

3 Hermite interpolation.

When using composite Gauss–Legendre quadrature of moderately high order, composite polynomial interpolation is stable in double precision arithmetic provided that the linear systems for the coefficients of the interpolating polynomials are solved in a backward stable way. Taking \( \tilde{\mu}(t) \) as the \( n - 1 \) degree polynomial interpolating the values \( \tilde{\mu}(t_j) \) on the panel where \( t \) is located may therefore be better than Nyström interpolation if high achievable accuracy is of concern. Unfortunately, the order of convergence is halved. We want to retain the convergence order of Nyström interpolation while fixing the stability problem. One idea is to choose \( 2n \) data points in the polynomial interpolation, \( n \) of which are picked from the panel where \( t \) is located and the remaining points come from neighbouring panels. This retains the convergence order, but the interpolating polynomial of degree \( 2n - 1 \) now must interpolate \( \tilde{\mu}(t_j) \) at several panels, increasing the error for a given mesh. A better idea is to seek an interpolating polynomial of degree \( 2n - 1 \) using data from the panel where \( t \) is located only. This can be achieved by Hermite interpolation: Let the two vectors \( \alpha \) and \( \beta \) have entries given by

\[
\alpha_j = r(t_j) - \tilde{\mu}(t_j), \quad \beta_j = \sum_{k=1}^{N} K(t_j, t_k) \tilde{\mu}(t_k) w_k, \quad j = 1, \ldots, N,
\]

where \( K(t, s) \) is the kernel \( K(t, s) \) differentiated with respect to \( t \). Then

\[
\tilde{\mu}(t) = r(t) - H(\alpha, \beta)(t),
\]
where $H$ is an operator that performs Hermite interpolation in $t$ based on data $\alpha_j$ and $\beta_j$ sampled at the $n$ points $t_j$ belonging to the panel on which $t$ is located.

There are at least three important differences between Nyström interpolation (1.4) and Hermite interpolation (3.2). The first difference is computational economy. For few points $t$, Nyström is cheaper. For many points $t$, Hermite is cheaper. The second difference is convergence. Both interpolations should have the same convergence order, but Nyström may have a smaller constant in front of the leading error term since it, contrary to Hermite, does not require that $K(t,s)$ is resolved in $t$. The third difference is stability. Hermite should be superior to Nyström, as discussed above.

4 Numerical examples.

We present numerical examples performed in MATLAB on a Sun Blade 100 workstation. The boundary and its parameterization are given by

\[
z(t) = (1 + 0.3 \cos 5t)e^{it}, \quad -\pi < t \leq \pi,
\]

both for the Laplace problem (2.1) and for the biharmonic problem (2.2). Composite $n$-point Gauss–Legendre quadrature with $m$ panels, equisized in parameter $t$, and $n = 16$ is used for all calculations except for reference solutions $\mu(t_j)$. These are computed at 1000 points using the composite trapezoidal rule and
stored in vectors $\mu_{\text{ref}}$ and $t_{\text{ref}}$. The linear systems are solved with the GMRES iterative solver [14] including a low-threshold stagnation avoiding technique [7] and a stopping criterion threshold of $10^{-16}$ in the relative residual. For resolved systems this threshold is reached in between 17 and 19 iterations. Direct computation is used for the matrix-vector products with kernel limits $K(t, s)$ and $K_t(t, s)$ for $s = t$ computed analytically. The Galimberti-Pereyra algorithm [5] is used for the Hermite polynomials. This algorithm costs between $6.5n^2$ and $10.5n^2$ FLOPs for $2n$ coefficients depending on how much precomputation is done. The value of $t$ on a quadrature panel along with the $n$ nodes $t_j$ on that panel are transformed into the canonical interval $[-1,1]$ prior to interpolation.

First take (2.1) with Dirichlet data given by

$$f(t) = \Re \left\{ \frac{1}{z(t) - 1 - i} \right\}.$$  

(4.2)

Numerical results for the estimated relative interpolation error $E(\tilde{\mu}, \mu_{\text{ref}}, t_{\text{ref}})$ of (1.3) are presented in Figure 4.1. One can see that the initial convergence is the same for Nyström and for Hermite interpolation. At about about 250 discretization points cancellation start to pollute the Nyström interpolation and its error grows in an irregular fashion, reflecting how close some discretization points are to interpolation points on the various grids. Actually, the convergence order, which is 32, is never reached. As for Hermite interpolation, it converges stably. The almost constant relative error beyond 400 points suggests that the

Figure 4.2: Same as in Figure 4.1 but for $\tilde{\mu}(t_j)$ of (2.2).
reference solution itself may not be accurate to more than 15 digits. Clearly, Hermite interpolation outperforms Nyström interpolation in this example.

Now take (2.2) with stress data given by

\begin{equation}
{h(t)} = -\frac{2z(t)z'(t)}{z'(t)},
\end{equation}

corresponding to a prescribed traction along the boundary of \(-2iz(t)z'(t)/|z'(t)|\).

Figure 4.2 shows a convergence pattern that is similar to that of (2.1) in Figure 4.1, but with the difference that the convergence with Hermite interpolation now sets in later than the convergence with Nyström interpolation. The reason being that the product \(K(t, s)\mu(s)\) in (2.2) is easier to resolve in \(s\) than in \(t\). Recall that the convergence of Nyström interpolation is controlled by resolution in \(s\) only, while the convergence of Hermite interpolation is controlled by resolution in \(s\) as well as in \(t\).

However, consider the variable substitution

\begin{equation}
\mu^{*}(t) = \mu(t)z'(t),
\end{equation}

in (2.2) followed my multiplication with \(z'(t)\) from the left. From a numerical viewpoint this corresponds to a diagonal similarity transformation and, since it does not alter the spectrum of the system matrix, does not much affect the
convergence properties of the GMRES iterative solver. The number of iterations needed to meet the stopping criterion differ at most by one. The substitution, however, simplifies the presentation of the equation

\[ \mu^*(t) - \frac{1}{\pi} \int_{-\pi}^{\pi} \left[ 3 \left\{ \frac{z'(t)}{z(s) - z(t)} \right\} \mu^*(s) + \frac{3}{(z(s) - z(t))^2} \overline{\mu^*(s)} \right] ds \]

\[ + \frac{i z'(t)}{A} \int_{-\pi}^{\pi} \Re \{ \overline{z}(s) \mu^*(s) \} ds = -\frac{z'(t)}{2\pi i} \int_{-\pi}^{\pi} \frac{(h(s) - h(t))z'(s)}{z(s) - z(t)} ds, \]

and, more importantly, the product \( K(t, s) \mu^*(s) \) is now at least as easy to resolve in \( t \) as in \( s \). The convergence should be identical for Nyström and Hermite interpolation. Figure 4.3 shows that this is indeed the case and that Hermite interpolation again outperforms Nyström interpolation.

REFERENCES